TOPIC 1: NUMBERS & ALGEBRA

Date April 16th, 2024

1.1 Numbers - Rounding

→ Common sets of Numbers:

$\mathbb{N} = \{0, 1, 2, 3, 4 \ldots\}$ natural set

$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \pm 4 \ldots\}$ integer set

$\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$ rational set

$\mathbb{R} =$ rational set + irrational set

$\mathbb{R} \setminus \mathbb{Q} =$ irrational set e.g. $\pi, e, \sqrt{a}$ $(\mathbb{I})$

↳ where a is not a square number

$\mathbb{Z}^+ = \{1, 2, 3, 4 \ldots\}$ positive integer set

$\mathbb{Z}^- = \{-1, -2, -3, -4 \ldots\}$ negative integer set

$\mathbb{Z}^* = \{\pm 1, \pm 2, \pm 3 \ldots\}$ non-zero integers

↳ i.e. $\mathbb{Z}^* = \mathbb{Z} - 0 = \mathbb{Z}^+ + \mathbb{Z}^-$

→ Intervals

$x \in [a, b]$ = $a \leq x \leq b$

$x \in]a, b[$ or $x \in (a, b)$ = $a < x < b$

$x \in]a, b]$ or $x \in (a, b]$ = $a < x \leq b$

$x \in [a, +\infty[$ or $x \in [a, \infty)$ = $x \geq a$

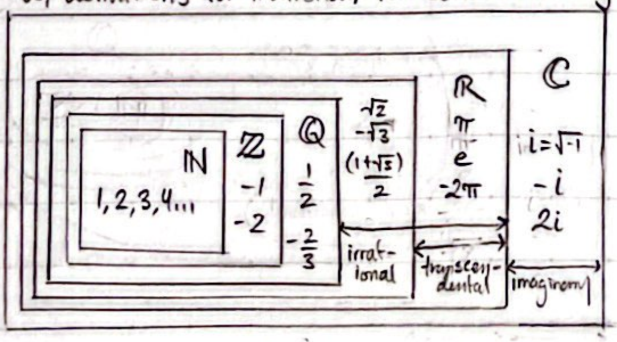
$x \in]-\infty, a]$ or $x \in (-\infty, a)$ = $x \leq a$

$x \in (-\infty a] \cup [b. \infty)$ = $x \leq a$ or $x > b$

* $x[$; $]x$, $]x[$ = $x)$, $(x$, (x)

→ Venn Diagram of Number Sets:

w/ definitions for irrational, transcendntal & img.



*ℂ will be discussed in its own dedicated section

→ Notation:

= equal to ≠ not equal to

< less than ≤ equal to or less than

> greater than ≥ equal to or greater than

{...} set of : such that

∈ element of ∉ not an element of

$[x, y]$ inclusive of x and y

(x, y) exclusive of x and y

$[x, y)$ inclusive of x, exclusive of y.

∪ union of ∩ intersection of

[=);] = (so $]a, b[= (a, b)$

→ Decimal Places, DP vs. Signifiam Figures, SF:

For the number, 123.4567, rounded to:

1 d.p = 123.5	Nearest integer:
2 d.p = 123.46	= 123
3 d.p = 123.457	Nearest 10:
1 s.f. = 100	= 120
4 s.f = 123.5	Nearest 100:
6 s.f = 123.457	= 100

→ IB Rounding Conventions:

3 significant figures (digits) → round such that everything past the third digit is a trailing zero.

If not 3 s.f. or explicitly stated otherwise, do exact to be more precise.

→ Scientific Notation, SN

$a \times 10^k$ where $1 \leq a \leq 10$, $a \in \mathbb{R}$, $k \in \mathbb{Z}$

$0.000012345 = 1.2345 \times 10^{-5}$

$123.45 = 1.2345 \times 10^2$

$\left.\begin{array}{l} \text{↳ k is the amount of leading 0s} \\ \text{↳ k is the number of digits after the} \\ \text{first digit for numbers} \leq 10 \end{array}\right\} |x|$

if $1 \leq x \leq 10$, it is already in SN

* The IB convention of 3 s.f. still applies so,

$123.45 \to 1.23 \times 10^2$, $0.000012345 \to 1.23 \times 10^{-5}$

1.2 Sequences in General - Series

→ Sequences

- A sequence is an ordered list of terms:

$= u_1, u_2, u_3, u_4, u_5 \ldots$

n:	1	2	3	4	5	6	7
Random numbers:	5	π	7	-3	$\sqrt{3}$	$\frac{1}{2}$	i
Odd numbers:	1	3	5	-7	9	11	13
Even numbers:	2	4	6	8	10	12	14
Multiples of 5:	5	10	15	20	25	30	35
Powers of 2:	2	4	8	16	32	64	128

$\overset{\nearrow (t_n)}{u_n}$ = term where n = the term's order

→ Series

- A series is the sum of terms expressed as:

$S_n = u_1 + u_2 + u_3 + \ldots + u_{n-1} + u_n$

$S_\infty = u_1 + u_2 + u_3 + u_4 + u_5 + \ldots$

S_n = a partial sum S_∞ = infinite sum

→ Sigma Notation: $\left(\sum_{n=1}^{k}\right)$

$u_1 + u_2 + u_3 + u_4 + u_5 + u_6 + u_7 = \sum_{n=1}^{7} u_n$

↗

the upper bound (=7) is often generalized = k where k is $\in \mathbb{N}$ and $k \notin 0$.

→ Generalizing u_n by General formula

$u_n = 2n$ gives $u_1 = 2, u_2 = 4, u_3 = 6 \ldots$

→ Generalizing u_n by a Recursive Relation (HL)

Given u_1, the first term; u_{n+1}, in terms of u_n

↳ first term & difference between terms in a series

$u_1 = 10$ $u_{n+1} = u_n + 2 \to u_2 = 10 + 2 = 12 \ldots$

At times both u_n and u_{n+1} are in the recursive series. A famous example is the Fibbonaci sequence:

$u_1 = 1$, $u_2 = 1$ $u_{n+1} = u_n + u_{n-1}$

↳ 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89...

1.3 Arithmetic Series (A.S.)

An arithmetic series is a sequence of numbers that has the difference of any consecutive numbers as a constant d. For instance:

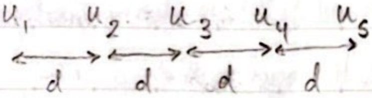
5, 8, 11, 14, 17, 20, 23, 26...

is an arithmetic sequence with a difference, d = 3 and a starting value, n_1 = 5 d can be any number but for IB will be constrained to be $\in \mathbb{Q}$.

→ Generalization of Arithmetic Series

$u_n = u_1 + (n-1)d$

This is true because:



Hence $u_5 = u_1 + 4d$

Similarly, $u_{10} = u_1 + 9d$, etc.

→ Sums of Arithmetic Series:

$S_n = \frac{n}{2}(u_1 + u_n)$ * u_n is known

$= \frac{n}{2}[u_1 + (u_1 + (n-1)d)]$

$= \frac{n}{2}[2u_1 + (n-1)d]$ * d is known

→ Consecutive Terms (AS.)

Let a, x, b be consecutive terms.

The common difference is equal to:

$x - a = b - x$, thus

is also equal to $2x = a + b \rightarrow x = \frac{a+b}{2}$

where x is the mean of a and b.

1.4 Geometric Series (G.S.)

A geometric series is a sequence that retains a constant ratio between terms.

For instance: 5, 10, 20, 40, 80...

is geometric because the common ratio, r is constant $(\frac{10}{5} = \frac{20}{10} = \frac{40}{10} = \frac{80}{40} = 2)$

→ Common Ratio

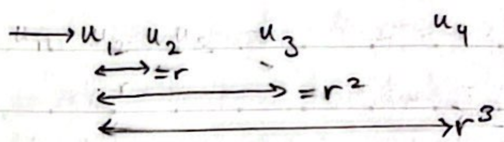
The common ratio, r is $\in \mathbb{Z}$, and can have alternating signs for its terms.

If $|r| < 1$, then the sequence will → 0.

→ Generalization of Geometric Series

$u_n = u_1 r^{n-1}$

This is because:



When the first term, u_1 is taken out as a coefficient, the remaining sequence is exponential, where the base is raised to the power of $n-1$.

→ Sums of Geometric Series:

$S_n = \frac{u_1(r^n - 1)}{r-1}$ or $S_n = \frac{u_1(1-r^n)}{1-r}$

Proof (HL):

$S_n = u_1 + u_1 r + u_1 r^2 + \ldots + u_1 r^{n-2} + u_1 r^{n-1}$ (1)

$r \cdot S_n = u_1 r + u_1 r^2 + u_1 r^3 + \ldots + u_1 r^{n-1} + u_1 r^n$ (2)

Hence (2)-(1) gives:

$r \cdot S_n - S_n = u_1 r^n - u_1$

$(r-1)S_n = u_1(r^n - 1)$

$S_n = \frac{u_1(r^n - 1)}{r-1}$ ☒

→ Consecutive Terms (G.S):

Let a, x, b be consecutive, geometric terms:

$\frac{x}{a} = \frac{b}{x} \rightarrow x^2 = ab \rightarrow x = \pm\sqrt{ab}$.

→ Sums of Infinite Geometric Series:

a) $S_n = \frac{u_1(r^n-1)}{r-1}$

if $n \to \infty$, then $r^n \to 0$ if $|r|<1$, thus:

$S_n \to \frac{u_1(0-1)}{r-1} = \frac{u_1}{1-r}$ OR ▨

b) $S_\infty = u_1 + u_1 r + u_1 r^2 + \dots$ ①

$rS_\infty = u_1 r + u_1 r^2 + u_1 r^3 + \dots$ ②

Assuming S_∞ exists, ①-②,

$S_\infty - rS_\infty = u_1 \Rightarrow S_\infty(1-r) = u_1$

$S_\infty = \frac{u_1}{1-r}$ OR ▨

c) $S_\infty = u_1 + u_1 r + u_1 r^2 + \dots$

$= u_1 + r(u_1 + u_1 r + \dots)$

$= u_1 + rS_\infty$

Assuming S_∞ exists,

$S_\infty = u_1 + rS_\infty$

$S_\infty - rS_\infty = u_1$

$S_\infty(1-r) = u_1$

$S_\infty = \frac{u_1}{(1-r)}$ ▨

1.5 Percentage Change - Financial Applications:

In problems with interest rates, population growth etc. r% is the rate of change, PV is the present value and there is a FV, future value, then:

$$FV = PV\left[1 + \frac{r}{100}\right]^n$$

This makes r into a decimal value (percent equivalent), then adds it to 1 to suggest a growth, r can be negative to express the PV decreasing.

* In financial calculations, IB uses 2d.p.

This is really just a geometric series where the common ratio R is $= 1 + \frac{r}{100}$

↳ IMO this is just easier to calculate w/out the finance app but here it is anyway:

TI-84 Plus CE: Apps → Finance → TVM Solver → Apps → Finance → tvm_ (variable solving for).

↳ keep PMT=0, P/Y = 1, C/Y=1.

→ Interest Compounded in k time periods:

If payments are made in years, n, and payments are made irreguarly, the formula applies:

$$FV = PV\left(1 + \frac{r}{100k}\right)^{kn}$$

For instance if k= 2, 2 payments would be made in a year, if $k = \frac{1}{2}$ a payment would be made every other year.

↓

C/Y = number of payments/year.

Investments with extra regular payments:

PMT = extra payments.

* Note that PMT and PV are negative if we are paying in the application.

→ Euler and Exponentiality:

Notice the frequency of payments ↑ & convergence of the FV as k↑. $n=5$, $PV=1000$, $r=12$

Yearly,	$k=1$	$FV = 1762$
Semiannually	$k=2$	$FV = 1791$
Quarterly,	$k=4$	$FV = 1806$
Monthly,	$k=12$	$FV = 1817$
Daily	$k=365$	$FV = 1822$
By the second	$k=31536000$	$FV = 1822$ ↓

This trend is the euler constant which is $= e \approx 2.718281828...$ and can be also expressed as:

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

→ Annuity, Amortization (HL):

Previously PMT, and PV were negatie now that payments are being recieved, they are positive. Here the last payment is ALWAYS include. This was not the case for payments.

Amortization = time taken to reach a value; this is typically when $FV=0$:

if $PV \times R^n - PMT\left(\frac{R^n-1}{R-1}\right) = FV$

then $PV \times R^n = PMT\left(\frac{R^n-1}{R-1}\right)$

→ Inflation: (Although theres a faster way, use:)

$$RV = \frac{FV}{(1+\frac{a}{100})^n}$$ where a is the inflation %.

→ How PV and PMT are Calculated (HL):

$$FV = PV \times R^n + PMT\left(\frac{R^n-1}{R-1}\right)$$

where $R^n = \left(1+\frac{r}{100}\right)^n$ and ↓ is extra.

* If the final payment is to be excluded subtrat 1 payment from PMT

1.6 The Binomial Theorem - $(a+b)^n$

$n! = n$ factorial $= 1\times2\times3\times...\times n-1\times n$

$1!=1$, $2!=2$, $3!=6$, $4!=24$, $5!=120$.

(for the sake of algebraic operations, $0!=1$)

→ Combinatorics

${}_nC_r$ or $\binom{n}{r}$ is read as "n choose r"

$${}_nC_r = \frac{n!}{r!(n-r)!} = \binom{n}{r}$$

For the 5 items, n how many combinations of 2 are there?

A, B, C, D, E

AB, AC, AD, AE, BC, BD, BE, CD, CE, DE

The answer is 10. *Order does not matter!

→ Binomial Theorem

$(a+b)^2 = a^2+2ab+b^2$

$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$

The coefficients of these expansions can be found in Pascal's triangle.

→ Pascal's Triangle:

By starting with $\overset{1}{1\ 1}$ and adding the two numbers next to each other (edges = 0), a pattern forms in Pascal's Triangle:

1 $n=0$
1 1 $n=1$
1 2 1 $n=2$
1 3 3 1 $n=3$
1 4 6 4 1 $n=4$
1 5 10 10 5 1 $n=5$
1 6 15 20 15 6 1 $n=6$
1 7 21 35 35 21 7 1 $n=7$
1 8 28 56 70 56 28 8 1 $n=8$

Formally each coefficient can be expressed as:

$$\binom{n}{r} \rightarrow (a+b)^5 = \binom{5}{0}, \binom{5}{1}, \binom{5}{2}, \binom{5}{3}, \binom{5}{4}, \binom{5}{5}$$
$$= 1, 5, 10, 10, 5, 1 \Rightarrow a^5 + 5a^4b + 10a^3b^2 + \ldots$$

In general a binomial expansion can be expressed as:

$$(a+b)^n = \binom{n}{0}a^nb^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n}a^0b^n$$

where the general term is expressed as:

$$\binom{n}{r}a^{n-r}b^r$$

If b xor a is negative alternate signs of terms.
↳ This will be expanded on with $n \in \mathbb{Q}$

1.7 Simple Deductive Proofs

A simple deductive proof is a sequence of independent operations made to premises such that two expressions are equivalent.

→ Proof of Equality (Identity)

Show that $A = B$ or $A \equiv B$

* $\equiv$ is a "check to see if equal" sign e.g.

$$(a+b)^3 \equiv a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)(a^2+2ab+b^2) \equiv a^3 + 3a^2b + 3ab^2 + b^3$$
$$a^3 + a^2b + 2a^2b + 2ab^2 + ab^2 + b^3 \equiv a^3 + 3a^2b + 3ab^2 + b^3$$
$$a^3 + 3a^2b + 3ab^2 + b^3 \equiv a^3 + 3a^2b + 3ab^2 + b^3$$

LHS (Left Hand Side) = RHS (Right Hand Side)

QED ▨

shows proof is finished.

Use operations to get A and B to be the same. Keep in mind that dividing variables is dividing by 0, ∴ not valid.

HL ONLY MATERIAL: TOPIC 1. ↓

1.8 Methods of Proof (HL) - Converse & contrapos.

Before other proofs see the following statements:

For a statement, if A then B;

- A converse statement is if B then A
- A contrapositive statement is if B then not A

↳ converse is not always true, contrapositive is.

e.g. OG. statement : if $x=1$, then $x^2=1$ ✓
converse : if $x^2=1$, then $x=1$ ✗
contrapositive : if $x^2 \neq 1$, then $x=1$ ✓

¬ and = or

¬ or = and

Date April 17th, 2024

→ Deductive Proof:

Let X be the name of a city

The usual process of reasoning is start from a hypothesis and reach the result by using logical steps.

Show that X is a Greek city, then X is a European city

Assume X is a Greek city

Greece is a part of Europe (subset)

Then X is a city within Europe

→ Proof by a Counterexample:

This is used to establish that a statement is not true. In general it is a converse s.

Let X be the name of a city

If X is a non-European, it is not a Greek city

Let X be a non-European city

Assume the result is false, i.e. X is a Greek city

But then X would be a European city.

This is a contradiction, thus X is not Greek.

→ Defining integers!

Even numbers = (2n), Odd numbers = (2n+1)

→ Proof by Counterexample:

"All cities in Europe are Greek"

Y is a city in Italy, — assume that Italy is in Europe

Therefore not all cities are Greek.

→ Pidgeonhole Principle:

A classic example of a proof by contradiction is:

Suppose that $n+1$ pidgeons are placed in n pidgeonhole, then there exist a pidgeonhole with at least 2 pidgeons (assuming $n \neq 0$)

1.9 Mathematical Induction (HL)

↳ used for statements, $P(n)$ in the form:

"For any $n \in \mathbb{N}$, it holds ... $P(n)$" or

"For any $n \geq 1$, it holds ... $P(n)$"

1. Show that the statement is true for $n=1$
2. Assume that the statement is true for $n=k$ (some k)
3. We prove that the statement is true for $n=k+1$ based on the assumption of step 2

Proof by induction that: $1+2+3+\ldots+n = \frac{n(n+1)}{2}$ fo

$$\text{LHS} = 1, \quad \text{RHS} = \frac{(1)(1+1)}{2} = 1$$

★ Assume that $1+2+3+\ldots+k = \frac{k(k+1)}{2}$

$$1+2+3+\ldots+k+(k+1) = (1+2+3+\ldots k)+(k+1)$$

$$\frac{(k+1)(k+2)}{2} = \frac{k(k+1)}{2}+(k+1)$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

QED.

Sometimes, we must use two steps in order to prove the next step. → 1. use $n = 1,2$.

↳ this applies to proofs for The Fibionacci sequ

e.g. Prove by induction: $u_n < 2^n$, $u_{n+1} < 2^{n+1}$

→ Useful statements:

Power of a number	a^n	$a^{k+1}=a^k\cdot a$
	a^{2n}	$a^{2(k+1)}=a^k\cdot a^2$
n factorial	$n!$	$(k+1)!=k!(k+1)$
	$u_1+u_2+\dots+u_n$	$=(u_1+u_2+\dots+u_k)+u_{k+1}$
Sum of terms	$\sum_{r=1}^{n}u_r$	$\sum_{r=1}^{k+1}u_r=\left(\sum_{r=1}^{k}u_r\right)+u_{k+1}$
n^{th} derivative	$f^{(n)}(x)$	$f^{(k+1)}(x)=[f^{(k)}(x)]'$
	$\frac{d^ny}{dx^n}$	$\frac{d^{k+1}y}{dx^{k+1}}=\frac{d}{dx}\left(\frac{d^ky}{dx^k}\right)$
$(2n^{th})$ derivative	$f^{(2n)}(x)$	$f^{(2k+2)}(x)=[f^{(2k)}(x)]''$

1.10 Systems of Linear Equations (HL)

2×2 System:

$a_1x+b_1y=c_1$

$a_2x+b_2y=c_2$

3×3 System:

$a_1x+b_1y+c_1z=d_1$

$a_2x+b_2y+c_2z=d_2$

$a_3x+b_3y+c_3z=d_3$

Two variables cal for two equations:

If the equations are equivalent, they overlap, thus having infinite solutions. If the 'slope' is the same and there is a different constant value, there are no solutions, else there is one solution.

Infinite solutions

↳ $0x=0$

no solutions

$0x=5$

finite solutions

$x=\frac{b}{a}$

* This is for 2D, 2 variables: solutions in 3D, 3 variables have lines as solutions... * → vectors

→ Gaussian Elimination

Take equations, label them, then multiply them by constants such that a variable is cancelled out when added

[There is no need to know augmented matrixes]

* Matrixes were not taught in class so omitted.

Variables can be isolated and substituted into another equation: $x+y=z$, ⇒ $z+y=2$ ⇒ $(x+y)+y=2$ ⇒ $2y+x=2\dots$

1.11 Complex Numbers - Basic Operations (HL)

$i^2=-1$, $\sqrt{-1}=i$, $\sqrt{-4}=2i$, $\sqrt{-5}=\sqrt{5}i$

For the equation: $x^2-4x+13=0$ ↘

In the quadratic equation $\Delta=-36$, thus there are no real solutions.

* $x=\frac{-b\pm\sqrt{|\Delta|}}{2a}$ for quadratic solutions where $\Delta=\ominus$, no real.

→ Complex Number definition:

$z=x+yi$ The real part, $Re(z)=x$

$\bar{z}=x-yi$ The imaginary part, $Im(z)=y$

→ Modulus:

Modulus argument of z shows the 'length' of the complex number: $|z|=\sqrt{x^2+y^2}$ for a complex number, $z=x+yi$.

$z = x+yi \quad \bar{z} = x-yi$
$-z = -x-yi \quad -\bar{z} = -x+yi$ } Same modulus.

→ Operations of Complex Numbers.

$z+w = (7+4i)+(2+3i) = 9+7i$

$z+w = (7+4i)-(2+3i) = 5+i$

$z+w = (7+4i)(2+3i) = 14+21+8i+12i$

$= 14+21+8i-12 = 2+29i$

$$\frac{z}{w} = \frac{(7+4i)}{(2+3i)} = \frac{(7+4i)(2-3i)}{(2+3i)(2-3i)} = \frac{14-21i+8i+12}{13}$$

$$= \frac{26-13i}{13} = 2-i \therefore \frac{7+4i}{2+3i} = 2-i$$

$|z|^2 = z \cdot \bar{z} \quad (x+yi)(x-yi) = x^2-y^2i^2 = x^2+y^2$

1.12 Polynomial Over the Complex Field (HL)

Fundemental Theorem of Algebra:

$f(x) = a(x-r_1)(x-r_2)$ → two roots

$f(x) = a(x-r_1)^2$ → two equal roots

$f(x)$ = irreductable → two complex roots.

In general all quadratics have 2 roots (in $\mathbb{C}$)

↳ A polynomial of degree $n > 1$ has n roots

If z is a solution, then $\bar{z}$ is also a solution*

↳ Because of this, solutions must compliment each other:

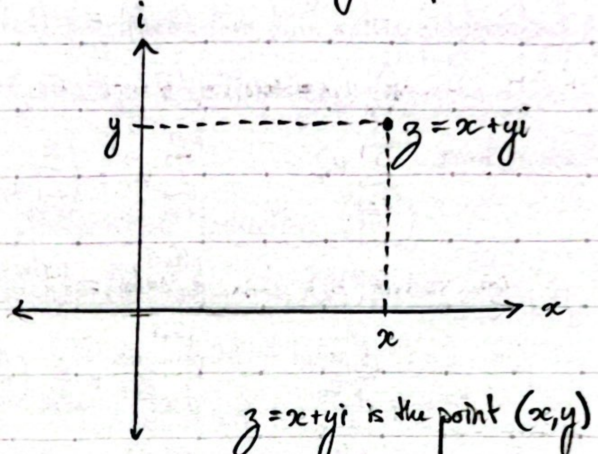
degree 3 functions can have: $3\mathbb{R}$; $1\mathbb{R}, 2\mathbb{C}$

degree 4 functions can have: $4\mathbb{R}$; $2\mathbb{C}, 2\mathbb{R}$, $4\mathbb{C}$

* odd functions must have a $\mathbb{R}$ solution.

1.13 The Complex Plane (HL)

The complex plane can be represented on the Cartesian Plane, where y is replaced with i.



$z = x+yi$ is the point (x, y)

Real part = x-coordinate, Imaginary = y-coordina

The modulus $|z| = \sqrt{x^2+y^2}$ is the distance from the origin. (Pythagorean Identity)

⊂ is subse

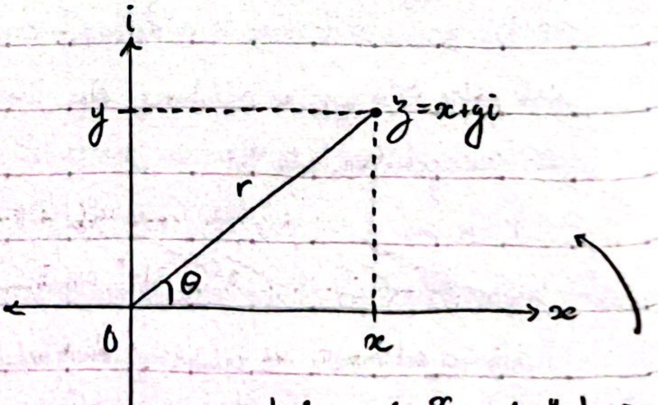
* $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

— $\bar{z}$ is symmetric to z about the x-axis

— $-z$ is symmetric to z about the y-axis.

→ Polar Form (Modulus Argument Form)

* The 'argument' is θ; think polar - think θ



$r = |z| \quad \cos\theta = \frac{x}{r}, \ \sin\theta = \frac{y}{r}, \ \tan\theta =$

For the principle argument, θ, we agree that
$-180° < \theta \leq 180°$ or $-\pi < \theta \leq \pi$

* Get into a habit of using radians

If $z = x + yi$, $x = r\cos\theta$, $y = r\sin\theta \longrightarrow$
$z = (r\cos\theta) + (r\sin\theta)i = r(\cos\theta + i\sin\theta)$

↳ cis form: $z = r\,\text{cis}\,\theta$ $(\text{cis}\,\theta = \cos\theta + i\sin\theta)$

→ Euler's Form

$z = re^{i\theta} = a+bi + r\,\text{cis}\,\theta = Re(z) + Im(z) = \arg(z)$

↳ Because this is not a trigonometric function, θ must be expressed in radians, $-\pi < x \leq \pi$

* this will be verified in calculus section

1.14 De Moivre's Theorem (HL)

Let $z = r\,\text{cis}\,\theta$. Then,

$z^{-1} = r^{-1}\text{cis}(-\theta)$ i.e. $\frac{1}{z} = \frac{1}{r}\text{cis}(-\theta)$ $\boxed{A}$

Remember $z \cdot \bar{z} = |z|^2 = r^2$ & $\bar{z} = r\,\text{cis}(-\theta)$

Then: $z^{-1} = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{r\,\text{cis}(-\theta)}{r^2} = \frac{1}{r}\text{cis}(-\theta)$

$=$

Let $z_1 = r_1\text{cis}\,\theta_1$ and $z_2 = r_2\text{cis}\,\theta_2$

$z_1z_2 = r_1r_2\,\text{cis}(\theta_1+\theta_2)$ $\boxed{B}$

$z_1z_2 = r_1r_2\,\text{cis}\,\theta_1\,\text{cis}\,\theta_2$
$= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$
$= r_1r_2[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 +$
$i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)]$
$= r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$
$= r_1r_2\,\text{cis}(\theta_1+\theta_2)$

$=$

Let $z_1 = r_1\text{cis}\,\theta_1$ and $z_2 = r_2\text{cis}\,\theta_2 \rightarrow$

$\frac{z_1}{z_2} = \frac{r_1}{r_2}\text{cis}(\theta_1-\theta_2)$ $\boxed{C}$

$\frac{z_1}{z_2} = z_1 z^{-1}$
$= r_1\text{cis}\,\theta_1 r_2^{-1}\text{cis}(-\theta_2)$ According to $\boxed{A}$
$= r_1r_2^{-1}\text{cis}(\theta_1-\theta_2)$ According to $\boxed{B}$
$= \frac{r_1}{r_2}\text{cis}(\theta_1-\theta_2)$

$=$

By $\boxed{B}$, the modulus of z_1z_2 is r_1r_2: $\overset{r_1r_2}{|z_1z_2|}$

By $\boxed{C}$, the modulus of $\frac{z_1}{z_2}$ is $\frac{r_1}{r_2}$: $\left|\frac{z_1}{z_2}\right| = \left|\frac{r_1}{r_2}\right|$

Thus by $\boxed{B}$, $\arg(z_1z_2) = \theta_1+\theta_2$, and $\arg\left(\frac{z_1}{z_2}\right) = \theta_1-\theta_2$ $\therefore$ $|z|$ preserves the operations, behaving $\arg(z)$ like $\log(z)$

→ De Moivre's Theorem:

$z^n = r^n\text{cis}(n\theta) = (re^{i\theta})^n = r^n e^{i\theta n}$

- For $n=1$ the statement is T: $z^1 = z = r\,\text{cis}\,\theta$
- Assume that $n=k$ is true for: $z^k = r^k\text{cis}(k\theta)$
- Claim: this is true for $k+1$: $z^{k+1} = r^{k+1}\text{cis}((k+1)\theta)$

hypothesis

$z^{k+1} = z^k z = r^k\text{cis}(k\theta)\,r\,\text{cis}\,\theta$
$= r^k r[\cos(k\theta) + i\sin(k\theta)][\cos\theta + i\sin\theta]$
$= r^{k+1}[\cos(k\theta+\theta) + i\sin(k\theta+\theta)] \longleftarrow \boxed{B}$
$= r^{k+1}[\cos(k+1)\theta + i\sin(k+1)\theta]$
$= r^{k+1}\text{cis}[(k+1)\theta]$ QED

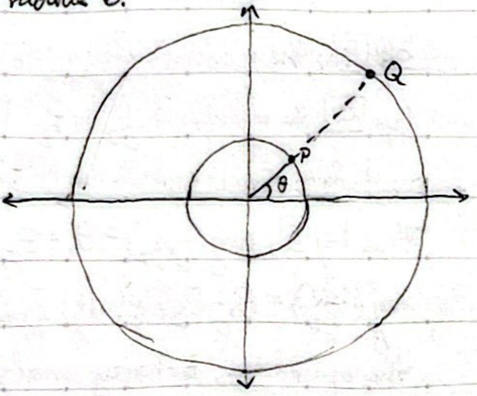
By mathematical induction, DeMoivre's Theorem is true for $n \in \mathbb{Z}$

→ Geometrical Interpretation of Multiplication

Let P be the point on the Complex Plane, which represents the following: $w = 2\text{cis}\theta$

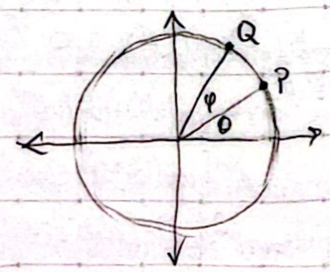
- P is on a circle of radius 2 (centered at O)
- The line OP forms an angle θ with x-axis

∘ If we multiply w by $z = 3$, the result is $wz = 6\text{cis}\theta$. The image point Q is on the circle of a radius 6.



= Enlargement by a factor scale of 3

∘ If we multiply w by $z = \text{cis}\,\phi$, the result is $zw = 2\text{cis}(\theta+\phi)$. The image line OQ forms an angle $\theta+\phi$ with x-axis. = Rotation with adding to angle, Im(z) and Re(z).



* Increasing r increases the magnitude of the numer (modulus of the complex #) Increasing θ rotates it counterclockwise.

1.15 Roots of $z^n = a$ (HL)

Equality in complex numbers is a strong relation: it gives a system of two equations.

$$x+yi = a+bi \Leftrightarrow \{x=a, y=b$$

$$r\text{cis}\theta = \rho\text{cis}\phi \Leftrightarrow \{r=\rho, \theta=\phi+2k\pi$$

→ n^{th} Roots of 1

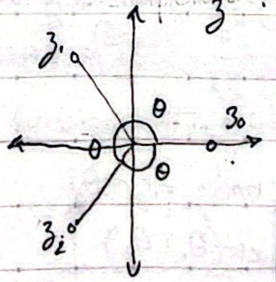
The solutions of $z^2=1$, otherwise the roots of the z^2-1 are 1 and -1. They are also known as the 2^{nd} roots of 1. → $z^n=1$ of the polynomial $y=z^n-1 \Rightarrow 0=z^n-1 \Rightarrow z^n=1$.

Let $z = r\text{cis}\theta$ be a root Then,

$$z^n=1 \Leftrightarrow r^n\text{cis}(n\theta)=1\text{cis}0 \Leftrightarrow \begin{cases} r^n=1 \Leftrightarrow r=1 \\ \theta=\frac{2k\pi}{n} \end{cases}$$

$$z_k = \text{cis}\frac{2k\pi}{n} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$$

→ Geometric interpretation



z^3: Find 1 solution and split the grid into n 'pieces' → $2\pi = n\theta$

$$\boxed{z_k = \sqrt[n]{\rho}\,\text{cis}\left(\frac{\phi+2k\pi}{n}\right.}$$

→ Factorization of z^n-1

If n is odd ~~or~~ then $(z-1)$ can be factored, and as 1 is a solution it is also a reference for division of θ. If even, $(z-1)(z+1)$ is a factor.

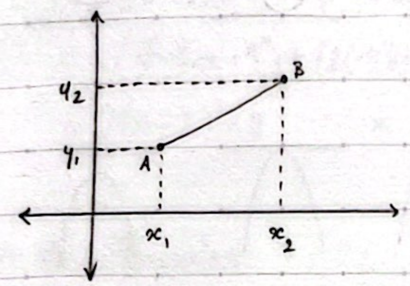
$z^n = a$

Topic 2: Functions

Date April 18th, 2024

2.1 Lines (Linear Functions)

→ Basic Notation on Coordinate Geometry:



The gradient (how much y is changing with respects to a change in x), slope is:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2-y_1}{x_2-x_1}$$

If the slope is 0, the line is horizontal

If the slope is positive, the line increases

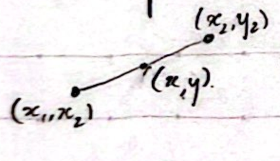
If the slope is negative, the line decreases

If the slope "is" infinite, the line is vertical

The distance between two points, A,B is:

$$d_{AB} = \sqrt{(\Delta x)^2-(\Delta y)^2} = \sqrt{(x_2-x_1)^2-(y_2-y_1)^2}$$

The midpoint between two points, A,B is:



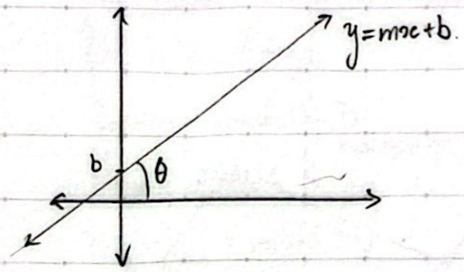
$$x = \frac{x_1+x_2}{2}, \quad y = \frac{y_1+y_2}{2}$$

Notation of a Function:

Linear: $y = mx+b$ or $f(x) = mx+b$

Quadratic: $y = ax^2+bx+c$ or $f(x) = ax^2+bx+c$

→ Equations of (linear, straight) lines



m = gradient/slope

* Vertical f(x) has the equation $x=c$ is ~~f~~ a function.

→ Parallel and Perpendicular Lines.

Parallel Lines have the slopes: $m_1 = m_2$ //

Perpendicular Lines have the slopes: $m_1 = \frac{-1}{m_2}$ X

Another way of defining a line is as:

$Ax+By=C$ where $B \neq 0$ (so the line is not vertical). If $A = 0$, the line is horizontal.

→ Given a Point and a Slope:

The line that passes through $P(x_0,y_0)$ w/ the slope, m.

$$\Rightarrow y-y_0 = m(x-x_0)$$

→ Given Two Points

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2-y_1}{x_2-x_1} \Rightarrow y-y_1 = m(x-x_1)$$

2.2 Quadratic Equations (Functions)

For the simplist quadratic, $y=x^2$,

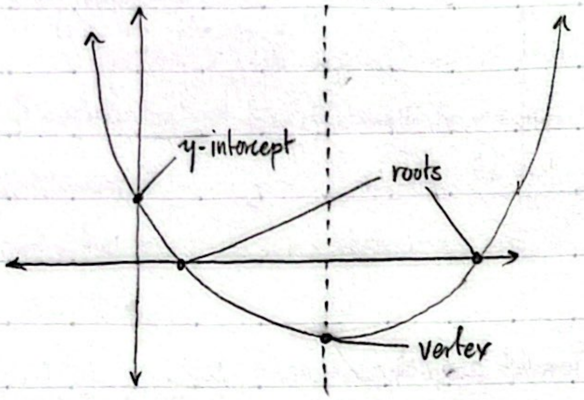
The Domain, of the function is $x \in \mathbb{R}$

The Range of the function is $y \geq 0$, $[0,+\infty)$

The 'curve' of the function is a parabola.

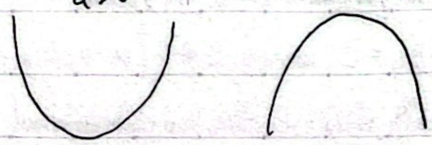
A quadratic function is given by:

$y = ax^2 + bx + c$ where:



1) $a \neq 0$, The sign of the coefficient of the largest degree variable determines:

$a>0$ $a<0$



concave up concave down

2) Discriminant: $\Delta = b^2 - 4ac$ → it determines the number of real roots:

$\Delta > 0$: 2 real roots, $\Delta = 0$: 1 real root

$\Delta < 0$: no real roots — only complex.

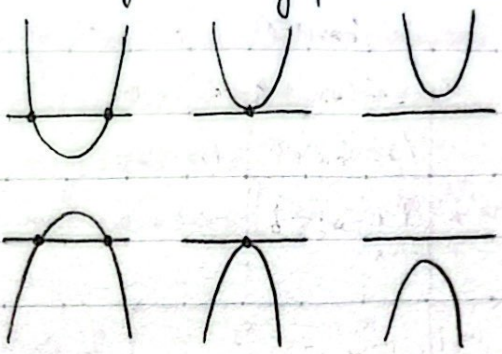
3) x-intercepts (roots): $x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}$ $\Delta \geq 0$

4) y-incercept: for $x=0$, we obtain $y=c$

5) Axis of symmetry: $x = \frac{-b}{2a}$ (also x coordinate of the vertex)

If we know the two roots, x_1, x_2, the vertex is at the mid point at: $(x = \frac{x_1+x_2}{2}, f(x))$

6) According to Δ, the graph looks like:



2 real roots $\Delta > 0$

1 real root $\Delta = 0$

No real roots $\Delta < 0$

→ Forms of a Quadratic Equation:

1) Traditional Form: $y = ax^2 + bx + c$

2) Factorization Form: $y = a(x - r_1)(x - r_2)$

3) Vertex-form: $y = a(x-h)^2 + k$

* r_1, r_2 are the roots; (h, k) is the vertex point.

The vertex coordinate's x-coordinate is given by:

$y = ax^2 + bx + c \longrightarrow x_v = \frac{-b}{2a}$

$y = a(x - r_1)(x - r_2) \longrightarrow x_v = \frac{r_1 + r_2}{2}$

Once x_v is found, find $f(x_v)$ using original form, then take $f(x_v) = k$ and put into the following:

$$y = a(x-h)^2 + k$$

Because k is the vertex it is one way to find the minimum or maximum of a quadratic.

Another method is the "completing the square" method = Factor out coefficients with x (a from a, a from b) then complete the square. E.g. →

$y = 2x^2 - 12x + 10$

divide by 2 then square for $\pm\left(\frac{b}{2}\right)^2$

$= 2(x^2 - 6x) + 10$

$= 2(x^2 - \boxed{6x} + 9 - 9) + 10$

$= 2(x^2 - 6x + 9) + (2(-9)) + 10$

$= 2(x-3)^2 - 8$

→ Vieta Formulas

$y = ax^2 + bx + c$, given the real roots, $r_1 r_2$:

S = sum of the roots $= r_1 + r_2$

P = product of the roots $= r_1 r_2$

The Vieta formula holds: $S = -\frac{b}{a}$, $P = \frac{c}{a}$,

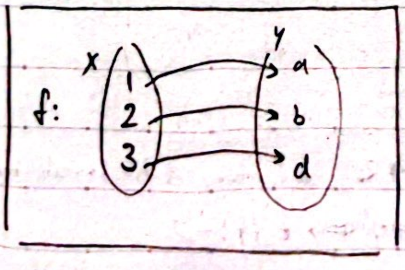
and that: $y = x^2 - Sx + P$

2.3 Functions, Domains, Range, Graph

→ Definitions

$f: X \to Y$ A function f form a set X to a set Y assaigns to each element x of X, a unique element y to Y.

We write:	We say:
$f(x) = y$	f maps x to y
$f: x \mapsto y$	y is the image of x



* X value must have 1 unique mapped Y value, however, distinct X vales can have the same Y

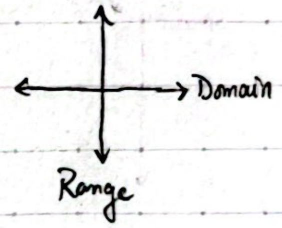
→ Domain & Range

For a function, f(x) the set of all the x values is the Domain and all of the possible y values, f(x) is the Range

Domain is denoted by, D

Range is denoted by R e.g.

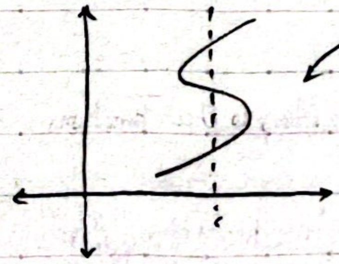
$D: x \in \mathbb{R}$, $R: y \in [0, 20]$



→ Graph

If every x value was mapped onto a Cartesian plane, corresponding to its respective y value, a graph is drawn. (*Ensure that restrictions on domains and range are respected).

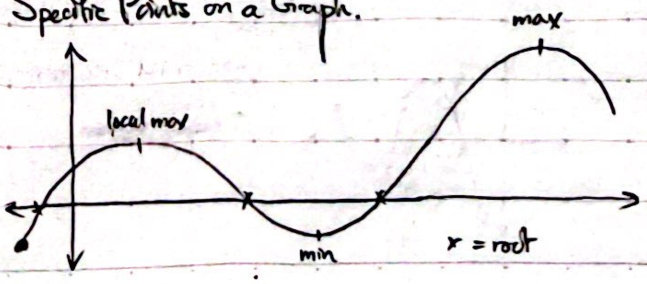
Vertical line test: If the graph visually has two y values it would 'overlap' with another point:

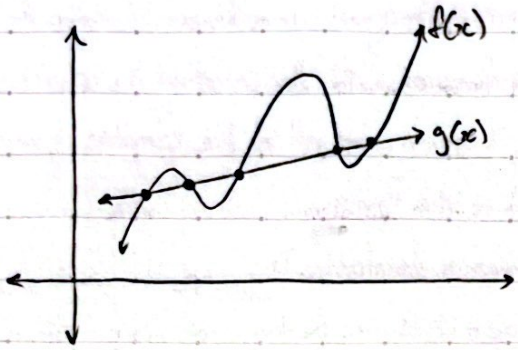


fails test ∴ not a function.

If the domain of a function is not given it is assumed that it encompasses all ∈ℝ

→ Specific Points on a Graph.

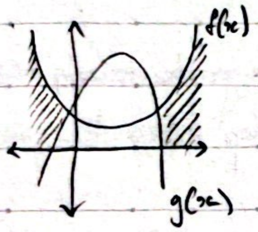




• = intersection points (where g(x) = f(x))

→ Solving Equations and Inequalities Using Graphs

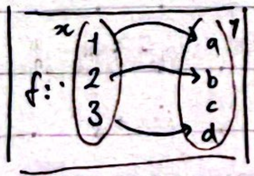
$y_1 = f(x)$ For solutions of $f(x) = g(x)$, the
$y_2 = g(x)$ x-coordinates are the intersections,
however, for $f(x) < g(x)$ intervals are used.



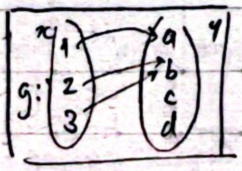
▨ = $f(x) > g(x)$

☐ = $f(x) < g(x)$

→ One to One vs. Many to One Functions

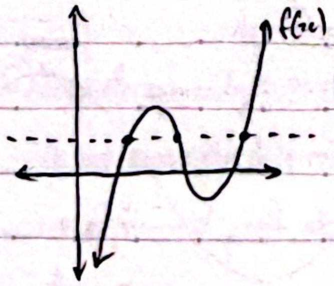


f(x) = one to one



g(x) = many to one

To check if a function is one to one:



Horizontal Line test
→ f(x) fails.

The formal definition of one to one is:

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$ or

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$ ← only case

2.4 Composition of Functions: $f \circ g$

Taking a function as a function:

$(f \circ g)(x) = f(g(x))$ = Composition

* $f \circ g \neq g \circ f$ it might but often doesn't.

→ Identity Function, $i(x)$

This is a function that maps x to itself:

$i(x) = x$, $i: \mapsto x$

* $(f \circ i)(x) = f(i(x)) = f(x)$

$(i \circ f)(x) = i(f(x)) = f(x)$.

→ Presupposition for $f \circ g$ and $g \circ f$ (HL)

Let $f: A \to B$ and $g: B \to C$ Then,

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, "g"] & C \\
x \arrow[r] \arrow[rr, bend right, "g \circ f"'] & f(x) \arrow[r] & g(f(x))
\end{tikzcd}$$

The domain of g(x) is constrained to the range of f(x)

$f \circ g$ g is applied first, then f $R_g \subseteq D_f$

$g \circ f$ f is applied first, then g $R_f \subseteq D_g$

2.5 The Inverse Function: $f^{-1}(x)$

Let $f: R \to R$, the inverse of this would be:

$$f(x) = y \Leftrightarrow f^{-1}(y) = x$$

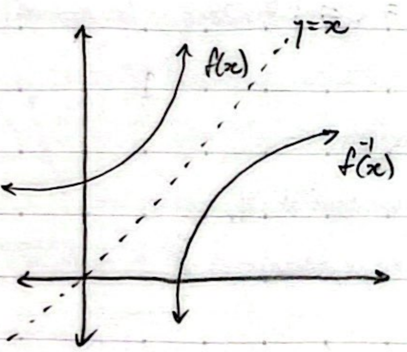
→ Finding $f^{-1}(x)$

1. Set $f(x)=y$ $x+10=y$
2. Solve for x $x=y-10$
3. Keep solution & replace y with x $f^{-1}(x)=x-10$

$$\left[\begin{array}{l} *(f^{-1})^{-1}(x)=f(x) \parallel D_{f^{-1}}=R_f \parallel R_{f^{-1}}=D_f \\ (f^{-1}\circ f)(x)=x=(f\circ f^{-1})(x))=i(x) \end{array}\right]*$$

$*(f\circ g)^{-1}=g^{-1}\circ f^{-1}$

→ Graph of $f^{-1}(x)$

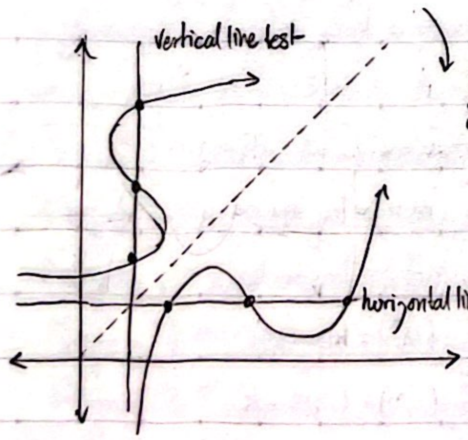


* The inverse of a function is symmetric to f about the line $y=x$

* Intersections occur at $f^{-1}(x)=f(x)=x$

→ Presupposition (mainly for HL)

If a function fails the horizontal line test, its inverse will fail the vertical line test:



graphical representation

* This is why quadratics have inverse equations ($\sqrt{x}$), that only go in one direction ±...

2.6. Transformations of Functions

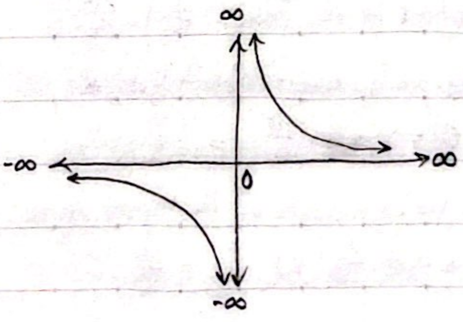
→ Basic transformations:

$f(x)+a$	Vertical translation 'a' units up	↑a	Vertical transformation
$f(x)-a$	Vertical translation 'a' units down	↓a	
$bf(x)$	Vertical stretch with a scale factor of 'b'	↕b	
$\frac{f(x)}{b}$	Vertical shrink with a scale factor of 'b'	↓↑b	
$-f(x)$	Reflection about the x-axis	↕	
$f(x+a)$	Horizontal translation 'a' units left	$\xleftarrow{a}$	Horizontal transformation
$f(x-a)$	Horizontal translation 'a' units right	$\xrightarrow{a}$	
$f(bx)$	Horizontal stretch with a scale factor of 'b'	$\xleftrightarrow{b}$	
$f(\frac{x}{b})$	Horizontal shrink with a scale factor of 'b'	→←b	
$f(-x)$	Reflection about the y-axis	↔	

When transforming graphs, take a key point: e.g. the vertex of a quadratic, min/max and transform the function relative to that point.

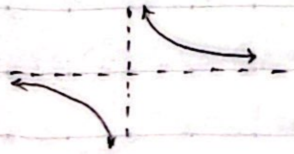
2.7 Asymptotes

For the function: $f(x)=\frac{1}{x}$:



As $x\to\pm\infty$, it approaches 0. = Horizontal asymptote.

As $x\to 0^{+}$ it approaches (→ from the righ) it approaches $+\infty$, as it $\to 0^{-}=-\infty$. This is the vertical asymptote.



Asymptotes are lines that the function cannot bisect, it acts as a 'barrier' or an unreachable value

In general, vertical asymptotes have equations of $x=a$, and horizontal asymptotes have $y=b$.

A vertical asymptote is a value that has no answer. For example $\frac{1}{x}$ divides by 0. Where as 0 is approached from the right, (e.g. +0.0...1) the y value is a very large positive number, $\to \infty$. However, when approached from the left, (e.g. -0.0...1), the y value is a very large negative number, $\to -\infty$.

A horizontal asymptote is the opposite, where no matter how large (in magnitude) a number gets, the converging value remains constant.

For equations in the form:

$$f(x) = a\left(\frac{1}{x-h}\right)+k,$$

horizontal asymptote

k determines the equation of the HA, and h determines the equation of the VA. (neg.)

→ Rational Functions in the Form: $f(x) = \frac{Ax+B}{Cx+D}$

VA = Find the points where $f(x)$ isn't defined:

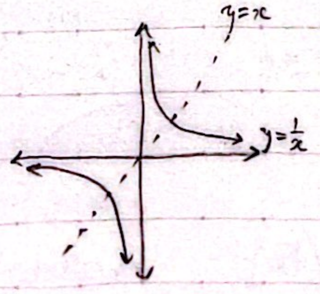
$$Cx+D=0 \Rightarrow x = \frac{-D}{C}$$

HA = Divide the coefficients of the highest degree:

$\frac{A(x)}{C(x)}$ as $x \to \infty$ find the ratio $= \frac{A}{C}$.

→ Self Inverse Functions

These are functions that are symmetric about the line $y=x$



2.8 Exponents - The Exponential Function / a^x

$a^m a^n = a^{m+n}$ $\quad$ $a^{-m} = \frac{1}{a^m}$

$(a^m)^n = a^{mn}$ $\quad$ $a^0 = 1$

$(ab)^n = a^n b^n$ $\quad$ $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

$\frac{a^m}{a^n} = a^{m-n}$ $\quad$ $a^{m+1} = a^m \cdot a$

→ The Number e

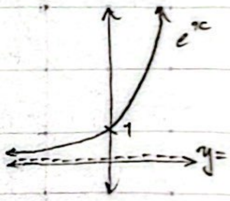
Recall from the instantaneous compound interest, the converging value that came from the followin

$$\left(1+\frac{1}{n}\right)^n = e \approxeq 2.7182818\ldots \text{ (as } n \to \infty)$$

for the function $y=e^x$

there is a y intercept at $y=1$,

a horizontal asymptote at $y=0$



2.9 Logarithms - The Logarithmic Function: $y=\log_a x$

The logarithm is the inverse opperation of an exponential function (follows presupposition of D,R

$$\log_n x = y \quad n^y = x$$

→ for $n=e$, log = ln (natural log) log written without a base, is assumed to be $\log_{10}$

→ Logarithm Laws (Basic)

$\log_b(m \cdot n) = \log_b m + \log_b n$ $\quad$ $\log_b 1 = 0$

$\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n$ $\quad$ $\log_b b = 1$

$\log_b(M^k) = k\log(M)$

$\log_b(b^k) = (1)\cdot k = k$

→ Change base law & misc:

$$\log_a b = \frac{\log_c b}{\log_c a} \qquad \log_{a^m}(n) = \frac{1}{m}\log_a n$$

2.10 Exponential Equations:

for $a^x = b$, there are the solutions:

$$x = \log_a b, \ \frac{\log b}{\log a}, \ \frac{\ln b}{\ln a}$$

→ Exponential Growth & Decay:

In several applications istantaneous, continuous growth is exponential. Thus it can be expressed similarly to financial models:

$$P = P_0 e^{kt}$$ where:

P = New value, P_0 = Principle value, e is eulers number, t is time, and k is the rate.

HL ONLY MATERIAL : TOPIC 2

2.11 Polynomial Functions (HL)

A polynomial function is an expression in the form:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x^1 + a_0 x^0$$

The degree of a function is n (or simply the same degree as the highest degree variable)

Degree 0 = Constant $y = ax^0 = a$

Degree 1 = Linear $y = ax + b$

Degree 2 = Quadratic $y = ax^2 + bx + c$

Degree 3 = Cubic $y = ax^3 + bx^2 + cx + d$

Degree 4 = Quartic $y = ax^4 + bx^3 + cx^2 + dx + e$

...

*When finding roots: ± factors of a_0, ± factors of $\frac{a_0}{a_n}$ where a_0 is a constant and a_n is the coeff of ↑ deg x

→ Division of Polynomials.

When $f(x)$ is divided by $g(x)$, there is a quotient, $q(x)$ and potentially a remainder, $r(x)$:

$$f(x) = g(x)q(x) + r(x)$$

where $r(x) = 0$ or $\deg r(x) < \deg g(x), q(x)$

From the Right to the Left Method.

- Step 1: setup:
- Step 2: Divide $2x^3$ by x^2
- Step 3: Multiply $2x$ by $g(x)$
- Step 4: Subtract.
- Step 5: Repeat

$$\begin{array}{r l}
 & 2x - 10 \\
x^2+3x+1 \,\big) & 2x^3 - 4x^2 + 5x - 1 \\
 & -\ \underline{2x^3 + 6x^2 + 2x} \downarrow \\
 & -10x^2 + 3x - 1 \\
 & -\ \underline{-10x^2 - 30x - 10} \\
 & 33x + 9
\end{array}$$

* Once degrees add to $\deg P(x)$

$$\therefore 2x^3 - 4x^2 + 5x - 1 = (x^2+3x+1)(2x-10) + 33x + 9.$$

→ Factor Theorem:

$f(x)$ is divisible by $(x-a)$, then $f(a) = 0$

↳ a is thus a root of $f(x)$

→ Remainder Theorem:

When $f(x)$ is divided by $(x-a)$, the remainder is a

→ Graphing a cubic function:

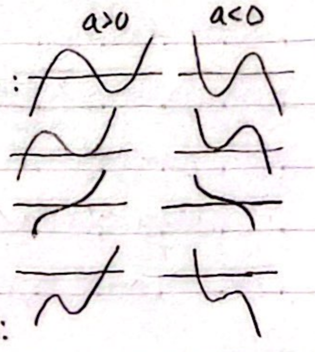
$f(x) = ax^3 + bx^2 + cx + d$

$a(x-r_1)(x-r_2)(x-r_3)$:

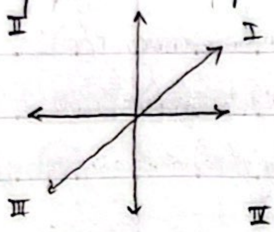
$a(x-r_1)^2(x-r_2)$:

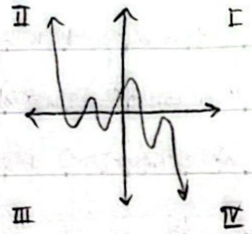
$a(x-r_1)^3$:

$a(x-r_1)(x^2-px+qx)$ irreducible:



* All odd functions have 1 root as they go from quadrant III to quadrant I or from II to IV

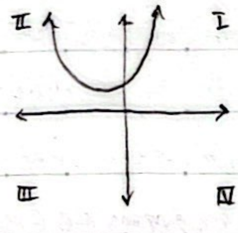


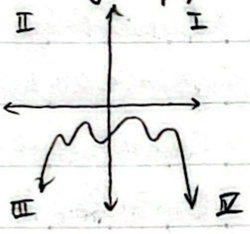


* This is as $x \to \pm\infty$, just a generalization

Because there is a bisection of I/II to III/IV, there has to be an intersection (polynomials)

This is not the case for even degree polynomials:





Because even polynomials go from II to I or III to IV, they don't have to bisect the x-axis as x approaches $\pm\infty$.

2.12 Sum and Product of Roots (HL)

$S = r_1 + r_2 + r_3 + \dots + r_{n-1} + r_n$ - Sum

$P = r_1 \cdot r_2 \cdot r_3 \cdot \dots \cdot r_{n-1} \cdot r_n$ - Product.

→ Vieta formula (applies to all polynomials)

for the form: $a_n x^n + a_{n-1} x^{n-1} + \dots a_1 x + a_0 x^0$

$S = \frac{a_{n-1}}{a_n}$ $P = (-1)^n \frac{a_0}{a_n}$

2.13 Rational Functions - Partial Fractions (HL)

A rational function has the form: $f(x) = \frac{p(x)}{q(x)}$

where $p(x)$ and $q(x)$ are polynomials.

* Recall asymptotes in the form: $\frac{Ax+B}{cx+D}$

→ Asymtotes of Rational Functions:

Vertical asymptotes are the roots of $q(x)$,

Horizontal asymptotes follow:

$\deg p(x) = \deg q(x)$ $y = \frac{\text{leading coefficient of } p(x)}{\text{leading coefficient of } q(x)}$

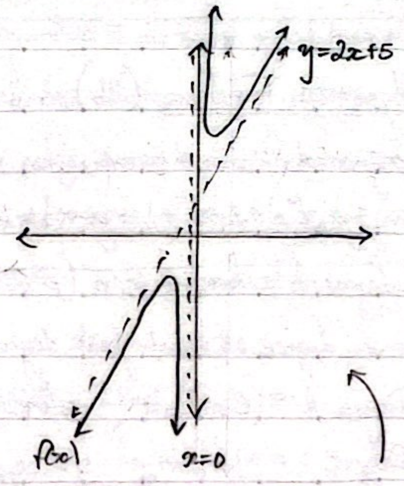
$\deg p(x) < \deg q(x)$ $y = 0$

$\deg p(x) > \deg q(x)$ there is no horizontal asympt

→ Oblique Asymptotes:

Rational functions in the form: $f(x) = \frac{ax^2+bx+c}{dx+e}$:

* Oblique asymptotes can have any degree but for IB they are only linear. Meaning that $\deg p(x)$ is 1 greater than $\deg q(x)$. This can be found to be the quotient of $p(x)/q(x)$ in the form $Ax+B$. (as the remainder will always be in the form $\frac{r}{dx+e}$ as $x \to \infty$ $\frac{r}{dx+e} \to 0$. (Thus is ommitted).



$f(x) = \frac{4x^2-2x+1}{2x-6} = 2x+5+\frac{31}{2x-6} \to 2x+5$ as $x \to \pm\infty$

→ Partial Fractions

* Only the simplist forms will be considered (Denominators with two roots):

$f(x)=\dfrac{a'}{ax^2+bx+c}$ and $f(x)=\dfrac{a'x+b'}{ax^2+bx+c}$

If the denominator has two roots, $x=r_1, r_2$ $f(x)$ can be expressed using partial fractions:

$$f(x)=\frac{A}{x-r_1}+\frac{B}{x-r_2}$$

for $f(x)=\dfrac{3x-5}{x^2-4x+3}$

$$\frac{A}{x-1}+\frac{B}{x-3}=\frac{A(x-3)+B(x-1)}{(x-1)(x-3)}=\frac{(A+B)x-(3A+B)}{(x-1)(x-3)}$$

from $f(x)$ $3x=(A+B)x$, $-5=-(3A+B)$

$$\therefore\ \left.\begin{aligned}A+B&=3\\3A+B&=5\end{aligned}\right\}\text{system of eq}\quad\begin{aligned}A&=1\\B&=2\end{aligned}$$

Thus, $f(x)=\dfrac{3x-5}{x^2-4x+3}=\dfrac{1}{x-1}+\dfrac{2}{x-3}$

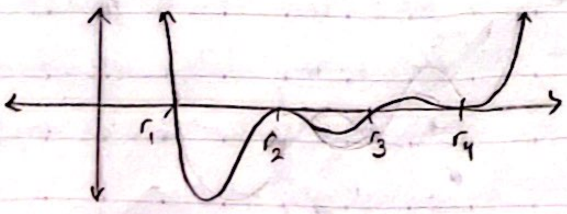
2.14 Polynomials and Rational Inequalities (HL)

Let $f(x)$ be a polynomial. By factorizing $f(x)$, a graph can be sketched and used to solve polynomial inequalities:

$$f(x)>0,\ f(x)<0,\ f(x)\geq 0,\ f(x)\leq 0$$

When $f(x)$ is factorized, linear factors and irreductable quadratic factors of the forms $(x-a)$ and (x^2+bx+c) [with $\Delta<0$], respectfully may be found: A single root (linear, odd degree) changes the sign, a double root (quadratic, even degree) doesn't change the sign:

For example: $f(x)=a(x-r_1)(x-r_2)^2(x-r_3)^3(x-r_4)^4$



* Assuming a is positive.

→ Finding Positive and Negative Regions:

1. Make a table with all the roots, and periods between as the table's columns
2. Make the rows as $f(x)$'s positivity / negativity:

e.g. $f(x)=(x-1)(x-2)(x-3)^2(x^2+x+1)$

x	$-\infty$		1		2		3		$+\infty$
$f(x)$		+	0	−	0	+	0	+	

2.15 Modulus Equations and Inequalities (HL)

Remember that a is a positive constant if:

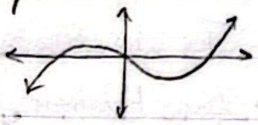
$$\left.\begin{cases}|x|=a \Leftrightarrow x=a \text{ or } x=-a\\ |x|<a \Leftrightarrow -a<x<a\\ |x|>a \Leftrightarrow x<-a, \text{ or } x>a\end{cases}\right\}*$$

In all three of these scenarios $|x|$ is positive

2.16 Symmetries of f(x) - More Transformations (HL)

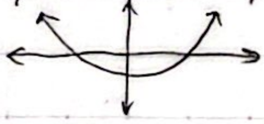
Odd functions are symmetric about the origin:

$f(-x)=-f(x)$



Even functions are symmetric about the y-axis

$f(-x)=f(x)$



* Symmetry is based on "what happens if $x \to -x$"

→ Absolute Value Transformations:

$f(x)\to|f(x)|$: All negative values become positive (if odd symmetry → even symmetry) = reflection about the x-axis.

$f(x)\to f(|x|)$ makes all x values negative = reflection about the y-axis. (for $x>0$)

Topic 3: Geometry & Trigonometry

→ The Reciprocal Function: $\frac{1}{f(x)}$

Another transformation on f(x) is the reciprocal function. Where if $a=x$ and a is a root of f(x), there is a VA at $x=a$ for $\frac{1}{f(x)}$. Similarly, if $x=a$ is a VA for f(x), then it is a root for $\frac{1}{f(x)}$. Any $y=a$ concept becomes $y=\frac{1}{a}$. To sketch this:

1) V.A becomes the roots and roots become VA
2) H.A $y=a$ becomes H.A. $y=\frac{1}{a}$
3) Any characteristic point (x,y) becomes $(x,\frac{1}{y})$
   ↳ local max/mins, y intercepts etc.
4) If f(x) is positive/negative, then so is $\frac{1}{f(x)}$
5) If f(x) is increasing, $\frac{1}{f(x)}$ is decreasing (& vice versa)

* There can be discontinuities, or non-permissable values, NPVs when taking the reciprocal.

← Previous Topic

3.1 Three Dimensional Geometry

→ 3D Coordinate Geometry:

Cartesian plane: $P(x,y)$, 3D Space: $P(x,y,z)$

The distance between two points $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$ is given by:

$$d_{AB}=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$$

and the midpoint of the resulting line segment:

$$M\left(\frac{x_1+x_2}{2},\ \frac{y_1+y_2}{2},\ \frac{z_1+z_2}{2}\right)$$

* These two formulas expand out with an increase in dimension degree. (2D → 3D)

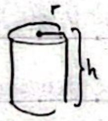
→ Volumes and Surface Areas of Known Solids:

$V=xyz$ $SA=2xy+2xz+2yz$

Cuboid

$V=\frac{1}{3}bh$ $SA=\sum$ areas of faces

Pyramid



$V=\pi r^2 h$ $SA=2\pi rh+2\pi r^2$

Cylinder

$V=\frac{4}{3}\pi r^3$ $SA=4\pi r^2$

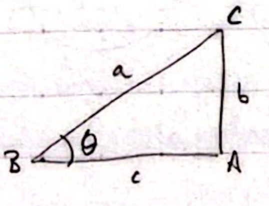
Sphere

* There are different 3D shapes but these are the basic forms; the V and SA for some may initially seem unintuative: this is because their formulas may be derived using calculus.

3.2 Triangles, Sine Rule, Cosine Rule

→ Basic Notations:

For any right angled triangle:

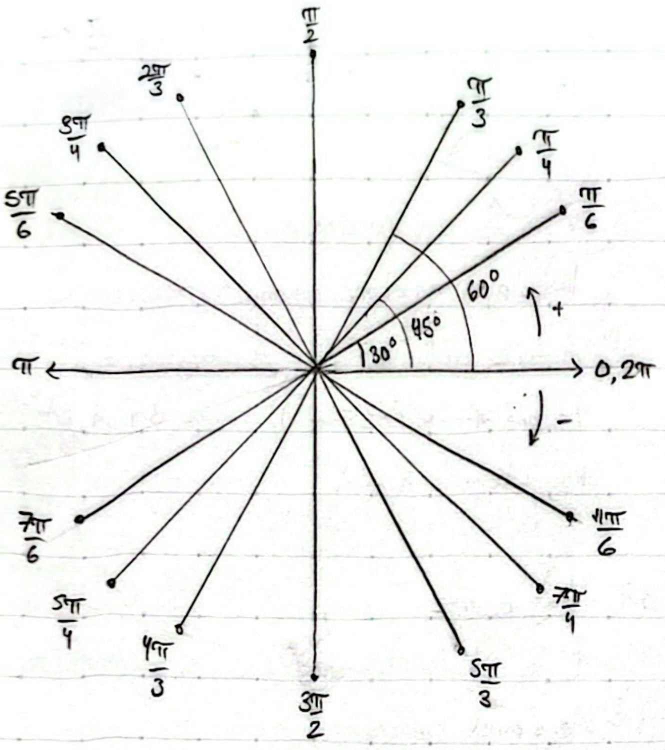


- $\sin\theta=\frac{b}{a}=\frac{\text{opposite}}{\text{hypotenuse}}$
- $\cos\theta=\frac{c}{a}=\frac{\text{adjascent}}{\text{hypotenuse}}$
- $\tan\theta=\frac{b}{c}=\frac{\text{opposate}}{\text{adjascent}}=\frac{\sin\theta}{\cos\theta}$

→ Pythagorean's Theorem: $a^2+b^2=c^2$; $\sin\theta^2+\cos\theta^2=1$

Angles A and B are complementary if $A+B=90°$

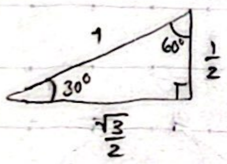
Angles A and B are supplementary if $A+B=180°$

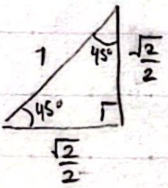


↗ π form, rad is preferred

$\pi = 180°$ so $30° = \frac{\pi}{6}$, $45° = \frac{\pi}{4}$, $60° = \frac{\pi}{3}$ etc.

The listed values have specific trigonometric ratios that can be found in to 30/60 & 45/45 triangles.





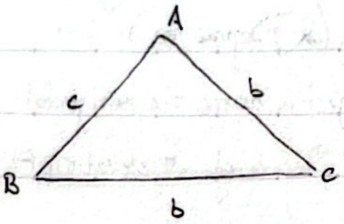
These triangles may be rotated, but the ratio persists:

	0,2π	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	DNE

* To convert from degrees to radians: $\frac{\theta°\pi}{180}$ rad $= \frac{180\theta}{\pi}°$

\+ Supplimentary angles have equal sines, but opposite cosines: $\sin 30° = 0.5$, $\sin 150° = 0.5$; $\cos 30° = \frac{\sqrt{3}}{2}$, $\cos 150° = -\frac{\sqrt{3}}{2}$

→ Sine Rule and Cosine Rule:



* For any triangle:

sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A$

↳ This can be rearranged for angle, A & b^2, c^2 as 'a^2'

↳ This is derived from the pythagorean theorem

→ Solution of a Triangle:

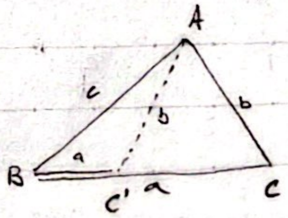
Any triangle has 3 sides and 3 angles: if

3 sides or 2 sides + 1 angle = cosine law

otherwise = sine law.

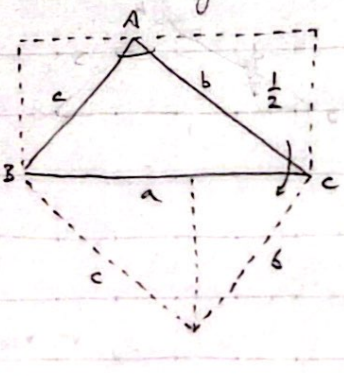
→ Ambiguous Case Triangles

When B, b and c are given, there are often 2 solutions:



* Use the sine law.

→ Area of a Triangle:



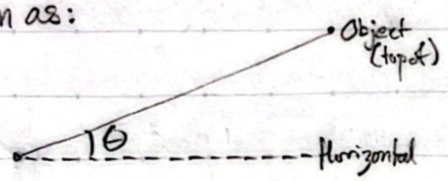
Area $= \frac{1}{2}bc\sin A$

* or variations where A → B, b → a etc.

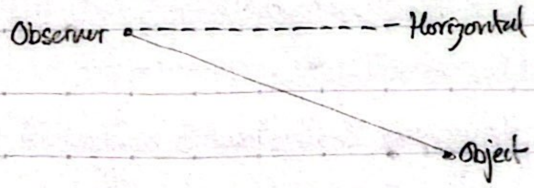
3.3 Applications in 3D Geometry - Navigation

→ Angle of Elevation (& Depression)

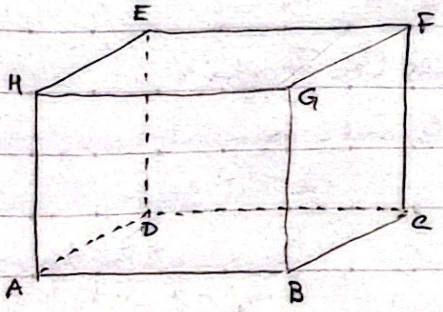
Suppose that an object is above the horizontal level of an observer. The angle of elevation, θ is shown as:



If this object is below the horizontal, it is the angle of depression, θ:



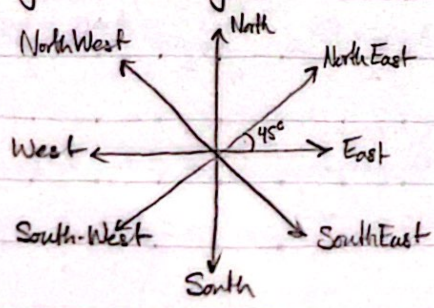
These notations are common for 3D shapes:

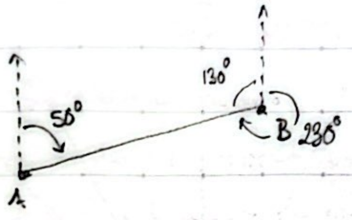


The angle of elevation from A to G is $B\hat{A}G$,

* The angle of depression from H to C is $F\hat{H}B$ etc..

→ Navigation: Bearings.





$180° - 50° = 130°$

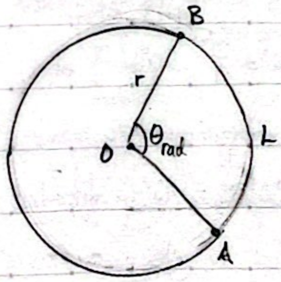
$360° - 130° = 230°$

In the diagram above, an object moves from A to B on the line, AB. A moves a w/ a bearing of 50° (AB = 50°), whereas B to A, BA has a bearing of 230°.

3.4 Unit Circle

* The actual unit circle was already drawn (oops!)

→ Arcs and Sectors

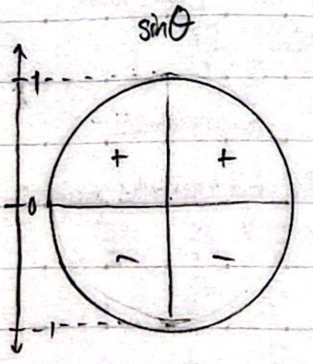


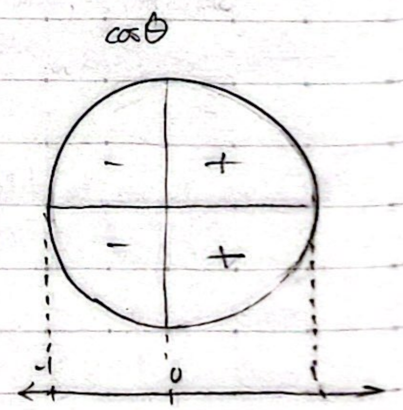
$L = r\theta$

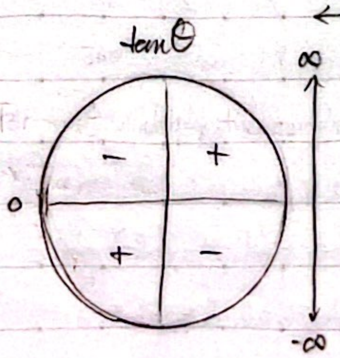
Area $OAB = \frac{1}{2}r^2\theta$

2π = full $2\pi \cdot \frac{1}{2}r^2 =$

formula of the circle: πr^2







* $\cos(\theta) = \cos(-\theta)$ ∴ eve

$\sin(-\theta) = -\sin\theta$ ∴ odd

$\tan(-\theta) = -\tan\theta$ ∴ odd

→ Trigonometric Identities:

Fundemental Pythagorean Identity:

$$\sin^2\theta + \cos^2\theta = 1$$

Double Angle Identities:

$$\sin2\theta = 2\sin\theta\cos\theta$$

$$\cos2\theta = \cos^2\theta - \sin^2\theta$$

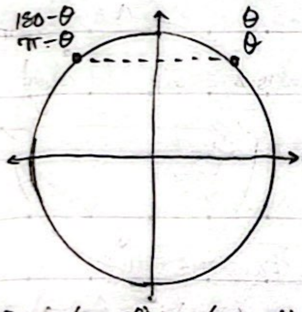
$$= 2\cos^2\theta - 1$$

$$= 1 - 2\sin^2\theta$$

$$\tan2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

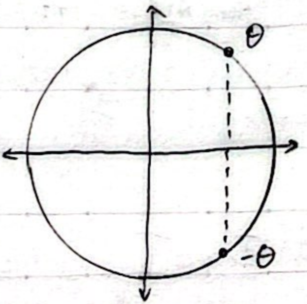
Derived from pythagorean, law of sines, law of cosines.

3.6 Trigonometric Equations



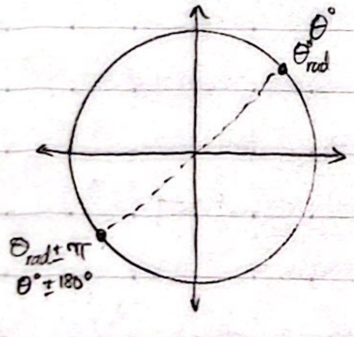
$\sin\theta = \sin(180-\theta^\circ) = \sin(180-\theta^\circ)$

↳ This because they have = y values.



$\cos\theta = \cos -\theta$

↳ same x values.

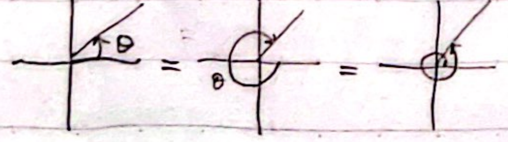


$\tan\theta^\circ = \tan(\theta^\circ \pm 180^\circ)$

$\tan\theta_{rad} = \tan(\theta_{rad} + \pi)$

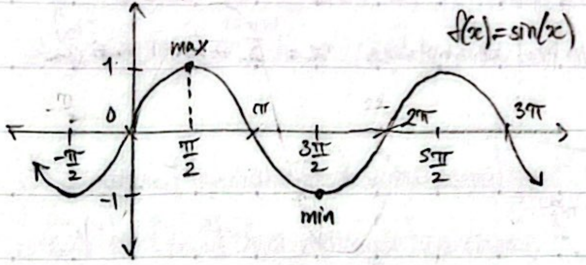
* This is with the additions of $\pm 2k\pi$ where $k \in \mathbb{Z}$, and exists to add rotations of θ with the same principle angle:

* Note this when making a general sol.



3.7 Trigonometric Functions.

→ $f(x) = \sin(x)$

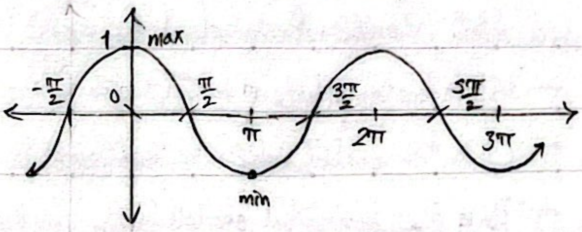


Domain: $x \in \mathbb{R}$, Range: $\{-1 \leq y \leq 1\}$

Central line: $y=0$, Amplitude: 1 (Δ between max and central line)

Period, T: 2π (length of a complete cycle)

→ $f(x) = \cos(x)$ (shifted sin(x) by $\pm \frac{\pi}{2}$)



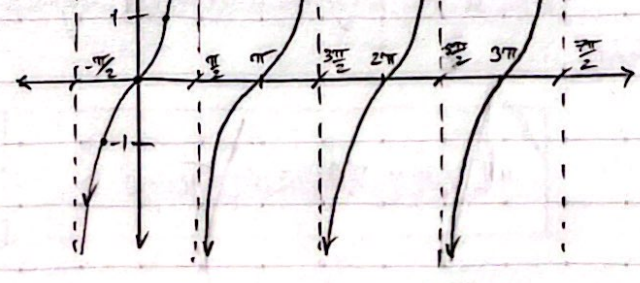
Domain: $x \in \mathbb{R}$ Range: $\{-1 \leq y \leq 1\}$

Central line: $y=0$ Amplitude: 1 Period, T: 2π

* Period can be calculated by finding the x-difference between max points (crests) or min points (troughs)

→ $f(x) = \tan(x) = \frac{f(x)}{g(x)}$ where $f(x) = \sin(x)$, $g(x) = \cos(x)$

x	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	…
$\tan(x)$	-	-1	0	1	-	



* There are roots where $\sin x = 0$, and VA where $\cos x = 0$

Domain: $x \in \mathbb{R}; x \neq \frac{\pi}{2} \pm 2k\pi, k \in \mathbb{Z}$ Range: $y \in \mathbb{R}$

(No minimum nor maximum), Period: $T = \pi$

Vertical asymptotes: $x = \frac{\pi}{2} \pm 2k\pi, k \in \mathbb{Z}$

→ Transformations of Sinusoidal Functions

A sinusoidal function is a function that can be expressed as $\sin(x)$ or $\cos(x)$.

These transformations follow conventional transformations of non transcendental $f(x)$s:

All $\sin(x)$ and $\cos(x)$ functions are generally:

$$f(x) = A\sin(Bx+D)+C$$

with $A>0$ (otherwise reflected about the x-axis):

- $|A|$ is the amplitude = vertical stretch factor
- C is the central value = vertical translation
- B is the horizontal stretch factor such that:

  Period, $T = \frac{2\pi}{B}$ or $B = \frac{2\pi}{T}$
- D is the horizontal translation magnitude

→ Transformations of $\tan(x)$

Similarly: $f(x) = A\tan(Bx+D)+C$

with $A>0$ (otherwise reflected about the x-axis):

- $|A|$ is the vertical stretch factor (NOT amplitude)
- $T = \frac{\pi}{B}$ is the period
- C is the new central line value
- D is the horizontal translation magnitude.

HL ONLY MATERIAL: TOPIC 3

3.8 More Trigonometric Identities and Equations. (HL

→ Reciprocal Trigonometric Functions:

secant: $\sec\theta = \frac{1}{\cos\theta}$ cosecant: $\csc\theta = \frac{1}{\sin\theta}$

cotangent: $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$

→ More trigonometric identities:

$\tan^2\theta + 1 = \sec^2\theta$ – derived from $\cos^2\theta$

$\cot^2\theta + 1 = \csc^2\theta$ – derived from $\sin^2\theta$

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$ (if + → +,)

$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$ (if + → −)

$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$ * if $A = B \Rightarrow$ Double ∧

→ More General Trigonometric Equations: (already cove

3.9 Inverse Trigonometric Functions. (HL)

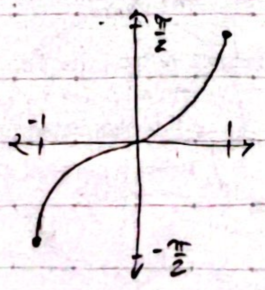
$\sin^{-1}x$ "undoes" $\sin x$ = $\arcsin x$

$\cos^{-1}x$ "undoes" $\cos x$ = $\arccos x$

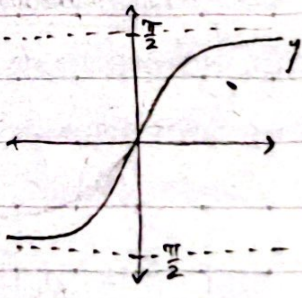
$\tan^{-1}x$ "undoes" $\tan x$ = $\arctan x$

* The domain and range are restricted for $\arcsin x$ and $\arccos x$. Arctan has a domain of $x \in \mathbb{R}$ and range of $y \in \{-\frac{\pi}{2}, \frac{\pi}{2}\}$. Keep in mind that inverse functions swap domains and ranges.

$\arccos x = \frac{\pi}{2} + \arcsin(-x)$



arcsin x



$y = \arctan x$

3.10 Vectors: Geometric Representation (HL)

→ Definitions

- Scalars = quantities with only magnitude

  e.g. age, temperature, length, time etc.

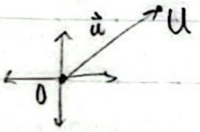
- Vectors = quantities with magnitude and direction

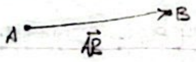
  e.g. force, velocity, acceleration, fields etc.

Geometrically Vectors can be expressed as:

a letter, $\vec{u}$ or two nodes: $\vec{AB}$

[A = tail, B = head]

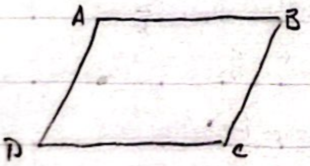




The length of the vector is the magnitude (scalar): $|\vec{u}|$ or $|\vec{AB}|$ and can be found with the Pythagorean Theorem: $\sqrt{(x_B-x_A)^2+(y_B-y_A)^2}+\dots$

→ Equal Vectors: $\vec{u}=\vec{v}$

Equal vectors have the same direction and magnitude (they can have different positions/coordinates).



$\vec{AB}=\vec{DC}$ $\vec{AD}=\vec{BC}$

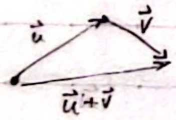
$\vec{BA}=\vec{CD}$ $\vec{DA}=\vec{CB}$

...

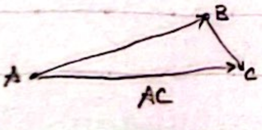
→ Addition of Vectors: $\vec{u}+\vec{v}$

To add two vectors add them from tip to tail:

...

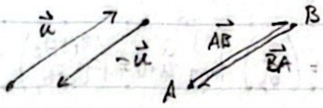


(This notation isn't often used)



$\vec{AB}+\vec{BC}=\vec{AC}$

→ Opposite Vector (Subtraction), $-\vec{u}$:



$^*\vec{AB}=-\vec{BA}$

$\vec{u}+\vec{v}=\vec{v}+\vec{u}$ Communicative Law

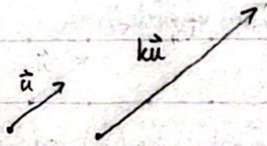
$\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$ Associative Law.

* $\vec{AB}=\vec{AC}+\vec{CB} \Rightarrow \vec{AB}=\vec{AC}+\vec{CD}+\vec{DE}+\vec{EB} \Rightarrow$ etc.

→ Zero Vector:

The zero vector is a vector with no magnitude nor direction: $\vec{u}-\vec{u}=0$, $\vec{AB}+\vec{BA}=0$, $\vec{AA}=0$ etc.

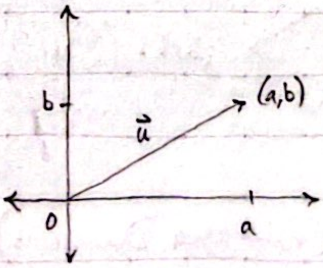
→ Multiplication by a Scalar:



where $k\in\mathbb{R}^+$ (to ensure the direction isn't flipped (this is done separately))

→ Vectors on a Cartesian Plane:

* All vectors can be moved to have their tails at the origin such that $\vec{AB}$ becomes $\vec{b}$. This makes calculating the magnitude easier as $x_A, y_A \to 0$ in $\sqrt{(x_B-x_A)^2+(y_B-y_A)^2+\dots}$.



$\vec{u}=OA=\begin{pmatrix}a\\b\end{pmatrix}$

where $|OA|=\sqrt{a^2+b^2}$

3.11 Algebraic Representations of Vectors (HL)

→ Addition of Vectors:

if $\vec{u} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ then $\vec{u}+\vec{v} = \begin{pmatrix} a_1+a_2 \\ b_1+b_2 \end{pmatrix}$

→ The Opposite Vector: $-\vec{u}$

if $\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ then $-\vec{u} = \begin{pmatrix} -a \\ -b \end{pmatrix}$

→ Multiplication by a Scalar:

If $k \in \mathbb{R}$ (scalar) and $\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ then $k\vec{u} = \begin{pmatrix} ka \\ kb \end{pmatrix}$

If $k > 0$ $\vec{u}$ has the same direction as $k\vec{u}$

If $k < 0$ $\vec{u}$ has the opposite direction as $k\vec{u}$

→ Length of $k\vec{u}$:

$|k\vec{u}| = |k||\vec{u}|$

→ The Unit Vector: $\hat{u}$ (unit vector of $\vec{u}$)

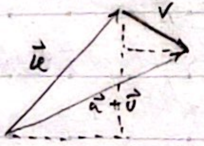
$$\hat{u} = \frac{1}{|\vec{u}|}\vec{u}$$

↳ has the same direction as $\vec{u}$ but has a magnitude of 1 unit.

→ The Notation: $\vec{u} = a\vec{i} + b\vec{j}$ $\vec{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

→ Connection Between Geometric and Algebraic Representations of Vectors:

* Take the x, and y components of vectors and the resulting algebraic opperations on all components is the algebraic representation:



$\vec{u}+\vec{v} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$

...

→ 3D Vectors (+ Additional Dimensions)

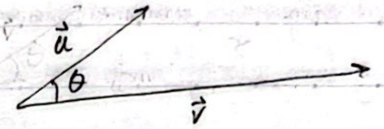
$\vec{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\vec{i} + b\vec{j} + c\vec{k}$ where $\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$|\vec{u}| = \sqrt{a^2+b^2+c^2 \ldots}$ etc. for $M, d_{AB} \ldots$

3.12 Dot Product - Angles Between Vectors (HL)

→ Geometric Definition:

Let $\vec{u}$ and $\vec{v}$ be two vectors, and θ be the angle between those two vectors:



The dot product (scalar) product of $\vec{u}$ and $\vec{v}$ is:

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$$

→ Algebraic Definition:

Let $\vec{u} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ be two vectors, and θ be the angle between those two vectors: The dot (scalar) product of $\vec{v}$ and $\vec{u}$ is given by:

$$\vec{u} \cdot \vec{v} = a_1 a_2 + b_1 b_2$$

→ Basic Properties of the Dot Product:

$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ Communicative Law

$\vec{u} \cdot (\vec{v}_1 + \vec{v}_2) = \vec{u} \cdot \vec{v}_1 + \vec{u}\vec{v}_2$ Distributive Law.

$k(\vec{u} \cdot \vec{v}) = (k\vec{u}) \cdot \vec{v} = \vec{u} \cdot (k\vec{v})$

→ Explanation for Algebraic Definition:

$\vec{u} \cdot \vec{v} = (a_1\vec{i} + b_1 j)(a_2\vec{i} + b_2\vec{j}) = a_1 a_2 \vec{i}^2 + a_1 b_2 \vec{i}\vec{j} + b_1 a_2 \vec{j}\vec{i} + b_1 b_2 \vec{j}^2$

$= a_1 a_2 \vec{i}^2 + b_1 b_2 \vec{j}^2 = a_1 a_2 + b_1 b_2$ ↑

$\left(i^2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} = 1 \ldots,\ ij + ji = 0 \right)$

→ Combination of Algebraic and Geometric Definitions:

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$$

→ Properties of Perpendicular, ⊥ and Parallel ∥ Vectors.

- $\vec{u} \perp \vec{v} \iff \vec{u} \cdot \vec{v} = 0$ (Perpendicular
- $\vec{u} \parallel \vec{v} \iff \vec{u} = k\vec{v}$ for some $k \in \mathbb{R}$ Parallel.

→ 3D Dot Product (+ Additional Dimensions)

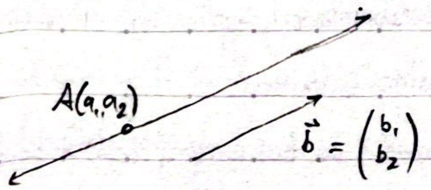
For two vectors $\vec{u} = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$,

the dot product is $\vec{u} \cdot \vec{v} = a_1a_2 + b_1b_2 + c_1c_2 \dots$

3.13 Vector Equation of a Line in 2D (HL)

Vector Equation: Let $A(a_1, a_2)$ be a point with the position vector: $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\vec{b}$ be a vector $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. There is a unique line that passes through A while being parallel to $\vec{b}$:

after exam do some visual explanation for some operations.



The position vector: $\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}$ of the random point, $P(x, y)$ is the line given by: $\vec{r} = \vec{a} + \lambda\vec{b}$ or $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ where λ is a parameter (shifts the magnitude, often time)

→ Parametric Equations:

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ gives $\begin{matrix} x = a_1 + \lambda b_1 \\ y = a_2 + \lambda b_2 \end{matrix}$

→ Cartesian Equation:

If λ is solved for: $\lambda = \frac{x - a_1}{b_1}$ and $= \frac{y - a_2}{b_2}$ $\therefore$

the relation between x and y is:

$$\frac{x - a}{b_1} = \frac{y - a_2}{b_2}$$

→ Given Two Points:

$A(a_1, a_2)$ and $B(b_1, b_2)$ are points, where the vector that encompases these two points, $\vec{AB}$ is equal to $\vec{b}-\vec{a}$, thus

$$\vec{r}=\vec{a}+\lambda\vec{b}$$

3.14 Vector Equations of a 3D Line:

Vector equation: $\vec{r}=\vec{a}+\lambda\vec{b}$ or $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}+\lambda\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$

Parametric equations: $x=a_1+\lambda b_1$, $y=a_2+\lambda b_2$ $z=a_3+\lambda b_3$.

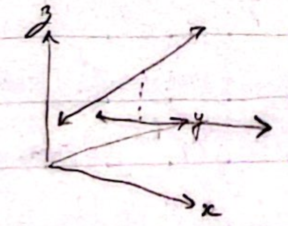
Cartesian equations: $\frac{x-a_1}{b_1}=\frac{x-a_2}{b_2}=\frac{x-a_3}{b_3}$.

→ Intersections of Lines.

Given the two lines $\vec{r}_1=\vec{a}_1+\lambda\vec{b}_1$, $\vec{r}_2=\vec{a}_2+\mu\vec{b}_2$ (2D) The lines will have an intersection, zero intersections or infinite. (if position and direction vectors are different; if position vector is different while direction vector is the same; and if both are the same then there are infinitely many solutions, respectfully. (or if the position vector is the same while the direction vector varries)).

The intersection of 2 (2D) vectors is found by setting $\vec{r}_1=\vec{r}_2$ then finding where $\lambda=\mu$ via a system of equations. (Sometimes the angle is asked for - use the dot product).

For 3D vectors, the lines may show: they don't intersect and aren't parallel.



3.15 Kinematics and Vectors (HL)

A common application of vectors is found in kinemati

→ Velocity & Speed.

Suppose a body is moving along a straight line, with a constant velocity and position given by:

$$\vec{r}=\vec{a}+t\vec{b}$$

Where $\vec{a}$ is the position of the body at $t=0$

$\vec{b}$ is the velocity of the body $(\vec{v})$ (m/s) + dir

$|\vec{b}|$ is the speed of the body (m/s)

t is the parameter (often time)

These can be in both 2D and 3D

3.16 Cross Product (HL):

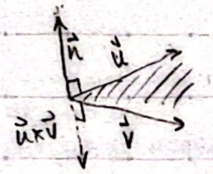
* This definition only applies for 3D vectors:

→ Geometric Definition.

Let $\vec{u}$ and $\vec{v}$ be two vectors, and θ be the angle between those two. where $0\leq\theta\leq\pi$. The cross product (vector product) is then the angle between those two vectors:

$$\vec{u}\times\vec{v}=(|\vec{u}||\vec{v}|\sin\theta)\vec{n}$$

Where $\vec{n}$ is the unit vector that is ⊥ to $\vec{u}$ and $\vec{v}$:



This is not communicative, however, $\vec{u}\times\vec{v}=\vec{u}\times-\vec{v}$

→ Algebraic Definition

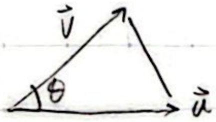
Let $\vec{u}=\begin{pmatrix}a_1\\b_1\\c_1\end{pmatrix}$ and $\vec{v}=\begin{pmatrix}a_2\\b_2\\c_2\end{pmatrix}$ be two separate vectors. The cross product is given by:

$$\vec{u}\times\vec{v}=\begin{pmatrix}b_1c_2-b_2c_1\\c_1a_2-c_2a_1\\a_1b_2-a_2b_1\end{pmatrix}$$

→ The Magnitude of $|\vec{u}\times\vec{v}|$

$\vec{u}\times\vec{v} = (|\vec{u}||\vec{v}|\sin\theta)\vec{n}$ implies that without the normal vector $|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$.

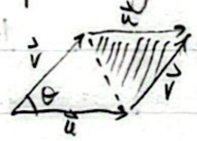
We know that the triangle determined by $\vec{u}$ and $\vec{v}$:



The area for such a triangle is given by:

$A_{\text{Triangle}} = \frac{1}{2}|\vec{u}\times\vec{v}|$ and when doubled gives the area of a parallelogram:

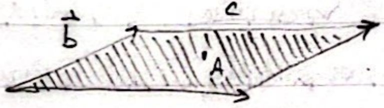
$A_{\text{parallelogram}} = |\vec{u}\times\vec{v}|$



3.17 Planes (HL)

→ Vector Equation of a Plane:

point: $A(a_1, a_2, a_3)$, and two vectors $\vec{b}, \vec{c}$.
Assuming $\vec{b} \nparallel \vec{c}$ a unique plane is formed (encompassing A)



The position vector, $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ of any point $P(x,y,z)$ of this plane is given by: $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c} =$

$$\vec{r}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \mu\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \text{ where}$$

λ and μ are parameters.

If $P(x,y,z)$ is any point on the plane, then AP lies on the plane determined by $\vec{b}$ and $\vec{c}$

$AP = \lambda\vec{b} + \mu\vec{c}$ (for some μ and λ), Then the position vector of P is given by:

$$\vec{r} = OP = OA + AP = \vec{a} + \lambda\vec{b} + \mu\vec{c}$$

→ Parametric Equations:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \mu\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \text{ gives: } \begin{aligned} x &= a_1 + \lambda b_1 + \mu c_1 \\ y &= a_2 + \lambda b_2 + \mu c_2 \\ z &= a_3 + \lambda b_3 + \mu c_3 \end{aligned}$$

→ Cartesian Equations: *

* μ and λ are eliminated, leaving:

$$Ax + By + Cz = D$$

This can be done by eliminating λ from the first two equations, then λ from the last two, then eliminating μ from the resulting two equations, e.g.:

Let $A(1,2,3)$ be the given point

$\vec{b} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ and $\vec{c} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$ be parallel vectors:

Then a plane passing through A, parallel to $\vec{b}$ & $\vec{c}$ is

$$\vec{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \mu\begin{pmatrix} 7 \\ 8 \\ 8 \end{pmatrix} \Rightarrow \begin{aligned} x &= 1 + 4\lambda + 7\mu \text{ ①} \\ y &= 2 + 5\lambda + 8\mu \text{ ②} \\ z &= 3 + 6\lambda + 8\mu \text{ ③} \end{aligned}$$

Eliminate λ from ① & ②

$5\times(1) - 4\times(2)$: $5x - 4y = 3 + 3\mu$ ④

Eliminate λ from ② & ③

$6\times② - 5\times③$: $6y - 5z = -3 + 8\mu$ ⑤

Next, eliminate μ from ④ & ⑤

$8\times④ - 3\times⑤$: $40x - 32y - 18y - 15z = -24 + 9$

$40x - 50y + 15z = -15$

Simplify to: $\boxed{-8x + 10y - 3z = 3}$

→ Vector Equations in Normal Form

Given a point, $A(a_1, a_2, a_3)$ and a normal vector, $\vec{n} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$, there exists a unique plane passing through A, perpindicular to $\vec{n}$.

The equation of such a plane is: $\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$

Indeed if $P(x, y, z)$ is a random point, then $AP \perp \vec{n}$. But $AP = OP - OA = \vec{r} - \vec{a}$, thus:

$$AP \cdot \vec{n} = 0 \Rightarrow (\vec{r} - \vec{a}) \cdot \vec{n} = 0 \Rightarrow \vec{r} \cdot \vec{n} - \vec{a} \cdot \vec{n} = 0$$

$$\Rightarrow \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

In the Cartesian form: $Ax + By + Cz = D$, the normal vector, $\vec{n} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$

3.18 Intersections Among Lines and Planes (HL)

→ Line/Line Intersection

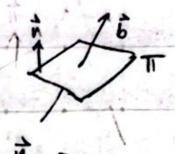
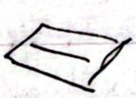
Line Relation:	Representation:	Check if...:
Parallel		$\vec{b}_1 \parallel \vec{b}_2$
Coincide		$\vec{b}_1 \parallel \vec{b}_2$ & $\vec{a}_1 = \vec{a}_2$
(One) Intersection		$\vec{r}_1 = \vec{r}_2$ has a solution
Skew		$\vec{r}_1 = \vec{r}_2$ has no solution

θ = the angle between $\vec{b}_1$ and $\vec{b}_2$: $\cos\theta = \dfrac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}$

→ Line/Plane Intersection:

Given Line: $L = \vec{r}_1 = a_1 + \lambda \vec{b}_1$ and the plane, $\Pi: Ax + By + Cz = D$ (so $\vec{n} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$), the following relationships and diagrams hold true:

$L_1: \vec{r}_1 = \vec{a}_1 + \lambda \vec{b}_1$, $L_2: \vec{r}_2 = \vec{a}_2 + \lambda \vec{b}_2$

Line & Plane	Representation:	Method:
Intersect at some point:		Plug $\vec{r}_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ into $Ax + By + Cz = D$ to find
Parallel		Check if $\vec{b} \perp \vec{n}$ or if there isn't an intersecti
Lies on the plane		Check if $\vec{b} \perp \vec{n}$ & at least 1 common point o if ∞ intersections/sol

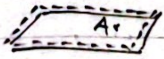
θ = The angle between $\vec{b}$ and $\vec{n}$ where $\theta = 90° - \phi$ $\qquad \sin\theta = \dfrac{\vec{b} \cdot \vec{n}}{|\vec{b}||\vec{n}|}$

* If lines are given in alternative forms, they must be transformed to $L: \vec{r}_1 = \vec{a}_1 + \lambda \vec{b}_1$, $\Pi: Ax + By + Cz = D$

→ Plane/Plane Intersection:

Given: $\Pi_1: A_1x + B_1y + C_1z = D_1$ so $\vec{n}_1 = \begin{pmatrix} A_1 \\ B_1 \\ C_1 \end{pmatrix}$

$\Pi_2: A_2x + B_2y + C_2z = D_2$ so $\vec{n}_2 = \begin{pmatrix} A_2 \\ B_2 \\ C_2 \end{pmatrix}$

Planes:	Representation	Method
Intersecting into a line: $\vec{r} = \vec{a} + \lambda \vec{b}$		Find two common points → and thus the line: common point, $\vec{a}$. direction vector, $\vec{b} = \vec{n}_1 \times \vec{n}_2$ or simultaneous eq.
Parallel		Check if $\vec{n}_1 \parallel \vec{n}_2$
Coincide:		Check if $\vec{n}_1 \parallel \vec{n}_2$ and if equations are multiples of each other (sharing A)

θ = angle between $\vec{n}_1$ and $\vec{n}_2 \Rightarrow \cos\theta = \dfrac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$

3.19 Distances (Self explanatory) (HL).

TOPIC 4: Statistics & Probability

4.1 Basic Concepts of Statistics:

Population: The entire list of a specified group

Sample: A subset (portion) of the population.

*Typically data is drawn from the population into a sample to make it easier to process:

→ Discrete vs. Continuous Data

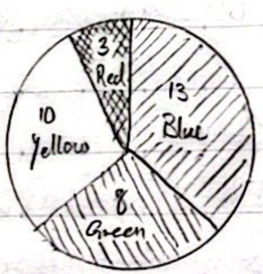
Discrete	or	Continuous
$\{10, 20, 30, 40\ldots\}$		$[40, 100]$
$\{-1, -2, -3, 0\ldots\}$		$\mathbb{R}$
(finite or numerable set)		(interval)

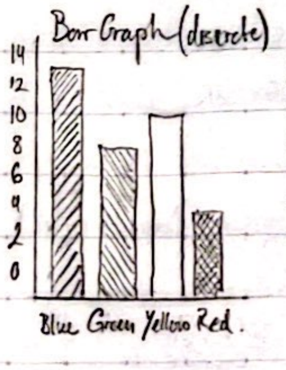
→ Representation of Data

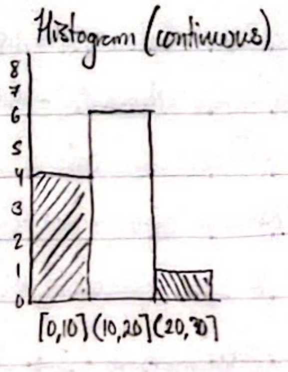
Frequency Table:

Color	Frequency
Blue	13
Green	8
Yellow	10
Red	3

Pie Chart:







Stem and Leaf Diagram:

Data:

12, 14, 16, 16, 20, 21

21, 21, 25, 32, 39, 40

43, 44, 47, 48, 49, 53

Stem	Leaf
1	2, 4, 6, 6
2	0, 1, 1, 5
3	2, 9
4	0, 3, 4, 7, 8, 9
5	3

→ Sampling with minimizing bias

For the population of 100,000, choosing a the following can be done to reduce bias.

- Simple Random Sampling:

  Select 1000 randomly where every element has an equal chance of being chosen.

- Systematic Sampling:

  As 100,000/1,000 = 100 (= period), pick a random number from 1 to 100, e.g: 20 and select the 20th person from each interval e.g 20th, 120th, 220th etc.

- Stratified Sampling:

  The population is divided into subgroups (people above/below 40 years) and then a sample is chosen

- Quota Sampling:

  As in stratified but we pick proportional samples according to the size of the subset relative to the overall population: Choose 1 person from <20 years 3 people from 20 years to 80 years and 1 final person from >80 years. *Assuming all intervals of 20 years have the same size: This is commonly used with normally distributed data.

Each method of sampling data has advantages/disadv.s

4.2 Measures of Central Tendency and Spread

Consider the following numerical data, L.:

10, 20, 20, 20, 30, 30, 40, 50, 70, 70, 80

The amount of terms (elements or entries) is $n = 11$

To relate the data to eachother 3 measure of central tendency and spread are taken:

→ Measures of Central Tendency (The 3 Ms)

Central tendency describes how the data behaves near some central value (data as a whole).

- Mean = Sum of all the values over n:

$$\text{mean} = \sum \frac{L_1[i]}{n} = \frac{10+20+20+20+30+30+40+50+70+70+80}{11} \boxed{= 40}$$

- Mode = The most frequent value:

  20 appears the most so mode = 20

- Median = The value placed at the middle: (provided the data is in ascending order),

  the median = $\lceil \frac{11}{2} \rceil = 6^{th}$ value $\lceil\ \rceil$ = ceiling

→ Measures of Spread:

Measures of spread describe the interrelation of data points, and how data is distributed.

- Standard Deviation, σ, s_n

  Standard deviation is the most "reliable" measure of spread as it takes all data into consideration: it measures how far data is from the mean.

- Range = (maximum value) - (minimum value)
- Interquartile Range = IQR = $Q_3 - Q_1$ where

  Q_1 = lower quartile = the median values before Q_2

  Q_3 = upper quartile = the median values after Q_2

  e.g: $10, 20, \underset{Q_1}{\textcircled{20}}, 20, 30, \underset{\text{medium}, Q_2}{\boxed{30}}, 40, 50, \underset{Q_2}{\textcircled{70}}, 70, 80$

  $IQR = Q_3 - Q_1 = 70 - 20 = 50.$

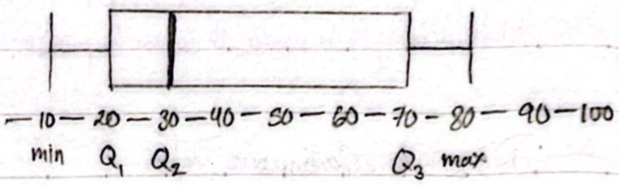
* Measures of Spread and Central Tendency are often calculated via (TI-84).

STAT → Edit → ↵ → L_1^* → 1-Vars → 2ND → 1.

→ Box Whisker Plot:

* $L_1 = 10, 20, \textcircled{20}, 20, 30, \boxed{30}, 40, 50, \textcircled{70}, 70, 80$

In an appropriate horizontal scale, mark min, Q_1, Q_2, $+Q_3$, ma



This diagram is useful in showing the density of data: 25% is lower than Q_1, 50% is between Q_1 and Q_3, above Q_2 is also 50% of the data (with their respective implicit inverses).

→ Percentiles

Q_1 = 25th percentile, Q_2 = 50th percentile, Q_3 = 75th ...

→ Outliers:

Extreme (extraneous) values skew the data and are considered outliers if they are:

below $Q_1 - 1.5 \times IQR$

above $Q_3 + 1.5 \times IQR$

→ More on Variance and Standard Deviation (HL)

If L_1 = a dataset of $x_1, x_2 \ldots x_n$, then,

variance is $\sigma^2 = \frac{\sum (x_i - \mu)^2}{n}$

standard deviation is $\sigma = \sqrt{\frac{\sum (x_i - \mu)^2}{n}}$

Variance measures the distance of each entry from the mean, μ, the squares of those distances, average of.

4.3 Frequency Tables - Grouped Data

The data, $L_1 = \{10,20,20,20,30,30,40,50,70,70,80\}$ can be expressed as a frequency table in the following:

Data, x	10,	20	30	40	50	70	80	n
Frequency, f	1	3	2	1	1	2	1	11

→ Mean with a Frequency Table, μ

$$\mu = \frac{f_1x_1 + f_2x_2 + f_3x_3 + \dots f_nx_n}{n} = \frac{\sum f_ix_i}{n}$$

→ Mode with a Frequency Table (x with greatest f)

→ Median with a Frequency (*sometimes cumulative) Table still $= \frac{n+1}{2}$ → just look for value # in f. table.

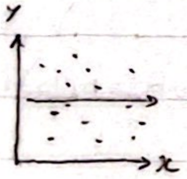
→ More on Varience (HL)

$$\sigma^2 = \frac{\sum f_i(x_i - \mu)^2}{n} \qquad \sigma = \sqrt{\frac{\sum f_i(x_i - \mu)^2}{n}}$$

4.4 Regression

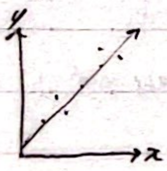
AKA Pearson const, coordinate

For scatter digarams (cartesian plots of data), the correlation coefficient?, r gives the relationship of the data, $-1 \geq x \geq 1$. Where the closer r is to ±1, the stronger the correlation is.



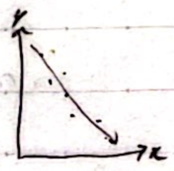
r=0,

no correlation; random spread



r>0

strong positive correlation



r<0

strong negative correlation

By using a trend line, values of y can be predicted for values of x not in the list.

* Values within the range, and are not datapoints are interpolations, values outside the range are extrapolations. In general, interpolations are more reliable that extrapolations.

→ Characteristics of a Regression Line.

- Regression lines pass through $M(\bar{x}, \bar{y})$ where the line, $y = mx + b$; the mean x value, $\bar{x}$, and the mean y value, $\bar{y}$.
- The data is seperated into almost two halves.

→ Finding Correlation on the TI-84 + CE:

Create a list: Stat → 1: Edit → Create L_1 → Stat → Calc → 4. LinReg (ax+b) if linear.

4.5 Elementary Set Theory.

→ Basic Notations:

A set is a collection of elements separated by commas. This set is often denoted by a capital letter such as ℝ, R, D, ℕ, ℚ etc.

- To declare a as an element of b: $a \in b$
- To declare f is not an element of b: $f \notin b$

- The set that contains no elements, {} or $\emptyset$

In general, if A contains n elements, there are 2^n subsets. E.g a set has 3 elements:

- P contains elements: A, B, C:

$\{\}, \{A\}, \{B\}, \{C\}, \{A,B\}, \{A,C\}, \{B,C\}, \{A,B,C\}$
= 8 subsets.

$\emptyset$ is always a subset $(\subseteq)$ of P

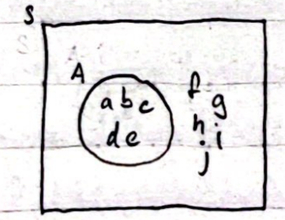
P is always $\subseteq$ of P

All subsets of P excluding P itself are also called proper subsets. To emphasize that B is a proper subset of A, we write:

$$B \subset A$$

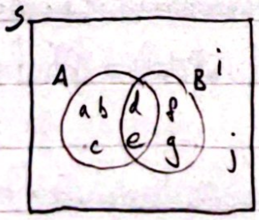
→ Venn Diagrams:

Large sets, S, are also refered to as universal sets: Let $S = \{a,b,c,d,e,f,g,h,i,j\}$ and $A = \{a,b,c,d,e\}$. A way to represent this is using a Venn Diagram:



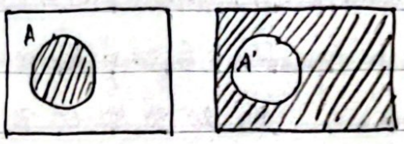
Venn Diagram of A as ⊂S.

If $B = \{d,e,f,g\}$, then:

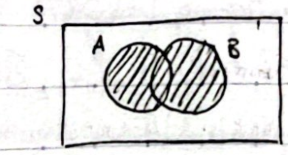


Venn Diagram of $A \cap B$; A, B as ⊂ S

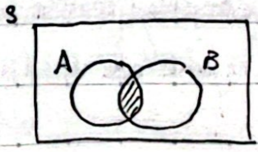
→ The Compliment of A: A', ¬A, not A, $\bar{A}$



→ The Union of A and B: A or B, $A \cup B$, A‖B



→ The intersection of A and B: $A \cap B$, A&&B

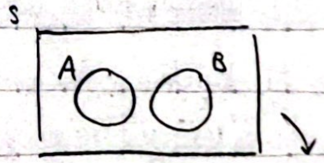


→ Basic Property (prevents double counting)

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

* $n(A)$ ≈ the # elements in A

→ Mutually Exclusive Sets: $A \cap B = \emptyset$, $n(A \cap B) =$



In this case only, $n(A \cup B) = n(B) + n(A)$.

4.6 Probability

The Universal Set, S is also known as the sample space containing all possible outcomes. Subsets are known as Events (of S).

The probability of A is denoted by:

$P(A) = \frac{n(A)}{\text{total}}$ (assuming all chances are =)

= if a random element is chosen from S,

→ Complimentary Events:

$P(A') = 1 - P(A)$

→ Combined Events

Recall: $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(given A and B aren't mutually exclusive).

If divided by the total the $P(A \cap B) =$

$P(A) + P(B) - P(A \cap B)$.

For mutually exclusive sets $P(A \cap B) =$

$P(A) + P(B)$.

★ Dice, Cards, and Tabular data are often used in questions.

4.7 Conditional Probability - Independent Events

$P(A)$ Probability of A

$P(A|B)$ Propability of A given B.

$$P(A|B) = \frac{n(A \cap B)}{n(B)} = \frac{P(A \cap B)}{P(B)}$$

*This is basically checking a portion of a subset as if the subset were the sample space.

↳ This is often questioned via tables and blunt statements of what n(A), P(A) etc. are.

→ Independent Events:

If $P(A|B) = P(A)$, B does not affect A; A is independant of B. Similarly if $P(B|A) = P(B)$, B is independant of A.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

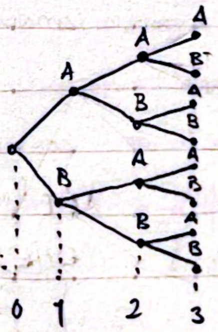
$$P(B)P(A|B) = P(A \cap B)$$

$$\longrightarrow P(A \cap B) = P(B) \cdot P(A)$$

4.8 Tree Diagrams

To represent a series of events dependent on another, tree diagrams are used to show dependence on several layers/depths:

Given A, A, A, A, B, B, B, B, B, B, there is a 0.4 chance of A, and a 0.6 chance of B. Assuming A, and B can be chosen again:



$AAA = (0.4)^3 = 0.064$

$AAB = (0.4)^2 \cdot (0.6) = 0.096$

$ABA = (0.4)^2 \cdot (0.6) = 0.096$

$ABB = (0.6)^2 \cdot (0.4) = 0.144$

$BAA = (0.4)^2 \cdot (0.6) = 0.096$

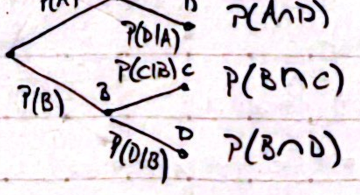
$BAB = (0.6)^2 \cdot (0.4) = 0.144$

$BBA = (0.6)^2 \cdot (0.4) = 0.144$

$BBB = (0.6)^3 = 0.216$

Total = 1.000

For the following:

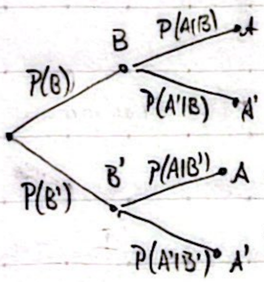


↙ This assumes that $A \cap B$ and that $C \cap D$ *both are independent of eachother

→ Bayès Theorem (HL)

"The reverse given" is what Bayes theorem is used for if $P(A|B)$, is given Baye's theorem outputs $P(B)$.

$$P(B|A) = \frac{P(B)P(A|B)}{P(B)P(A|B) + P(B')P(A|B')}$$



$P(A \cap B) = P(B)P(A|B)$

$P(A' \cap B) = P(B)(A'|B)$

$P(A \cap B') = P(B')(A|B')$

$P(A' \cap B') = P(B')(A'B')$

If there is three disjoint events, $B_1, B_2 B_3$ then the denominator becomes:

→ $P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B)_3$

4.9 Discrete Distributions in General

Generally speaking a random variable, X takes a random value within a given domain.

It can be discrete: or continuous:

$X \in \{10, 20, 30, 40\}$ $X \in [10, 40], 10 \leq X \leq 40$

$X \in \{0, 1, 2, 3, 4 \ldots\}$ $X \in \mathbb{R}$

→ Discrete Random Variables:

In general for a discrete random variable:

$$\begin{array}{c|ccc} x & x_1 & x_2 & x_3 \ldots \\ P(X==x) & p_1 & p_2 & p_3 \end{array}$$

it holds that

1. $p_1 \geq 0$ for all i
2. $\sum p_i = 1$ i.e. $p_1 + p_2 + p_3 + \ldots = 1$

→ Expected Value: $\mu = E(X)$

The mean $\overset{\mu}{}$ or otherwise, the expected value, $E(X)$ is defined by:

$$E(X) = \sum x_i p_i = x_1 p_1 + x_2 p_2 + x_3 p_3 + \ldots x_n p_n.$$

→ Median-Mode: (analogous)

Mode = The value $X = a$ of the highest probability

Medium = The value $X = m$ where the distribution/probability splits into two equal parts.

→ Variance. (HL)

Variance, is defined as the squre of the difference each value has from the mean:

$$Var(X) = E(X - \mu)^2$$

$$Var(X) = (x_1 - \mu)p_1 + (x_2 - \mu)p_2 + (x_3 - \mu)p_3 + \ldots$$

(or) $Var(X) = E(X^2) - \mu^2$

where $E(X^2) = x_1^2 p_1 + x_2^2 p_2 + x_3^2 p_3 + \ldots$

4.10 Binomial Distribution

The binomial distribution is a discrete distribution of the random $X \in \{0, 1, 2, 3, \ldots n\}$ with the distribution function:

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad x = 0, 1, 2, 3, \ldots$$

↳ This is not in the syllabus; only used in calc pr

Only two outcomes exist:

Success: probability p   Failure: probability p

given n attempts, and p = chance of success. while X counts the number of possible success

$$X \sim B(n, p)$$

This is expected to be performed by calculator. For TI-84+CE : 2nd → Vars (Distr) → A: binompdf(or B: p binomcdf → trials: n → p: p → x value: how many successes → paste. pdf = possibility of x, cdf = cummulity distribution to x:

$B_{pd}(x) = \text{binompdf}(n,p,x) = \text{EXACT}$

$B_{cd}(x) = \text{binomcdf}(n,p,x) = \text{CUMULATIVE}$

→ Expected Value and Variance of x

$E(X) = np \quad Var(X) = np(1-p)$ Then (HL);

$Var(X) = E(X^2) - E(X)^2$

$E(X^2) = Var(X) + E(X)^2$

$\underline{E(X^2) = np(1-p) + np^2}$ ★ Important.

→ Mode (HL)

◦ Check for expected number, if it is a decimal check the probability for neighboring values. The larger one is the mode: ← $n=20$, $p=\frac{1}{6}$

e.g: $P(X=3) = 0.237 > P(X=4) = 0.202$, $x=3$ is the mode.

If $E(X)$ is whole check neighboring integers:

$n=60$ $p=\frac{1}{6}$ so that $E(X) = \frac{60}{6} = 10$

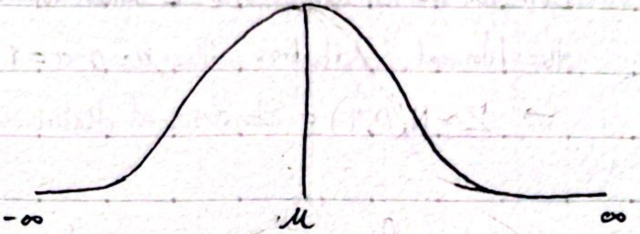
$P(X=9) = 0.134$, $P(X=10) = 0.137$, $P(X=11) = 0.126$

Hence the mode is 10.

4.11 Normal Distribution − $N(\mu, \sigma^2)$

A normal distribution is that of a continuous random variable, X with values from −∞ to ∞. Where μ is the mean and σ^2 = standard deviation.

The behaviour can be described by a function that looks like:

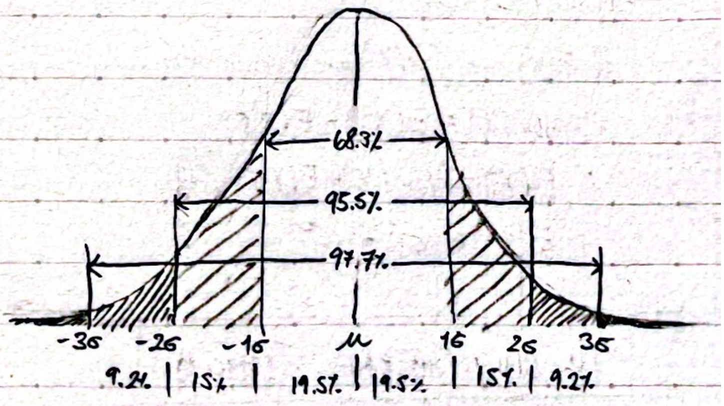


$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

↖ Not curriculum

Generally, there is a high chance that data will be distributed symmetrically about the mean, and there is a dramatic decrease as x deviates. A normal distribution can be described as:

$$X \sim N(\mu, \sigma^2)$$



→ Using a TI-84+CE

2nd → Vars (Distr) → 1: normalpdf((for one value) or 2: normalcdf((for asked for probability) or 8: invNorm((when probability is known). → enter x-value, μ, σ (and upper/lower bounds) → Paste.

→ Standardization - Normal Distribution, $N(0,1)$

Consider the random variable, Z which follows the Normal distribution with $\mu=0, \sigma=1$:

→ $Z \sim N(0,1)$ = Standardized distribution.

Any X that follows a normal distribution can be transformed into a standardized distribution by using the following formula:

$$Z = \frac{X-\mu}{\sigma}$$

Think of it as a horizontal shrinking and horizontal shifting to $x=0$ & the HC

→ Notice (HL)

If $E(X)$ and $Var(X)$ are known then...

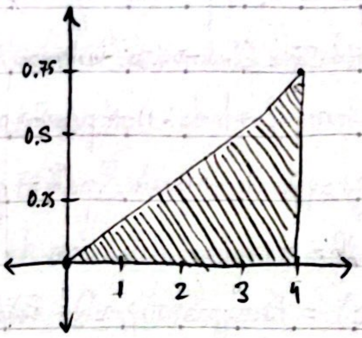
$$Var(X) = E(X^2) - E(X)^2$$
$$E(X^2) = Var(X) + E(X)^2$$
$$E(X^2) = \sigma^2 + \mu^2$$

HL ONLY MATERIAL: TOPIC 4

4.12 Continuous Distributions in General (HL)

Let X be a variable from 0 to 4, $[0,4]$. Suppose that X is not "uniformly" distributed throughout this interval, but it is more likely X obtains values as $X \to 4$. Assume that this function is $f(x)=\frac{1}{8}x$, $0 \leq x \leq 4$.

$*f(x) \geq 0$, $\int f(x)dx = 1$:



$0 \leq x \leq 4$

$f(x) = \frac{1}{8}x$

$\int f(x)dx = 1$

$* P(0 \leq X \leq 2) = 0.25$, $P(2 \leq X \leq 4) = 0.75$

$\int_0^2 \frac{1}{8}x\,dx = 0.25$, $\int_2^4 \frac{1}{8}x\,dx = 0.75$.

Only cdf is used, the only fixed value used is $P(X=a)=0$. ∴ it is also agreed that:

$$P(a \leq X \leq b) = P(a < X < b)$$

In the previous example:

$$f(x) = \frac{1}{8}x \quad 0 \leq x \leq 4$$

$$\therefore \int_{-\infty}^{\infty} f(x)dx = \int_0^4 \frac{x}{8} = \frac{x^2}{16}\Big]_0^4 = 1-0=1$$

→ Expected Value: $E(X)$

$$\mu = E(X) = \int_{\infty}^{\infty} x \cdot f(x)dx$$

→ Varience: $Var(X)$

$$Var(X) = E(X^2) - \mu^2$$
$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

For our example:

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^4 \frac{x^3}{8}dx = \frac{x^4}{32}\Big|_0^4 = 8$$

thus:

$$Var(X) = E(X^2) - \mu^2 = 8 - \frac{8}{3} = \frac{16}{3}$$

→ Discrete and Continuous $E(X)$, $Var(X_r)$, μ

X Discrete:

$\mu = E(X) = \sum x_i p_i$

$E(X^2) = \sum x^2 p_i$

X Continuous

$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$

$Var(X) = E(X^2) - \mu^2$

→ Mode = maximum of function

→ Medium $= \int_{-\infty}^{m} f(x) dx = 0.5$

→ Quartiles.

$\int_{-\infty}^{Q_1} f(x) dx = 0.25$, $\int_{-\infty}^{Q_2} f(x) dx = 0.5$

$\int_{-\infty}^{Q_3} f(x) dx = 0.75$ $\int_{-\infty}^{Q_4} f(x) dx = 1.0$.

4.13 Counting Permutations and Combinations (HL)

→ Multiplication Principle: Box Technique

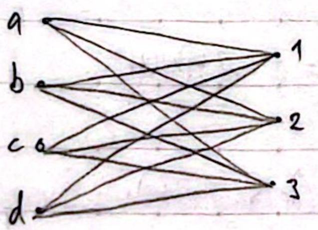
Suppose that

Task A has 4 outcomes: a, b, c, d

Task B has 3 outcomes: 1, 2, 3

The outcomes for A and B can be found by:

o. finding the total edges in:



a1, a2, a3
b1, b2, b3
c1, c2, c3
d1, d2, d3

Total = 12

Or by finding n(A) = 4 × n(B) = 3 = 12

In general, m choices for A, and n choices for B result in m×n total choices.

→ Words of r letters:

Consider an alphabet of n letters. Then number of words of r length $= n^r$

$\underline{n}\ \underline{n}\ \underline{n} \ldots \underline{n}$ (r times) $= n^r$

as n can be repeated chances multiply exponentially

→ Rearrangement of n objects:

Consider 5 objects: A, B, C, D, E how many permutations are there of n = 5?

$_nP_r$ if $n = r$ $_nP_r = r!$ (e.g. $5! = 120$).

→ Permutations and Combinations:

Consider n objects. How many ways are there to select them from n?

Combinations, $_nC_r$ (order doesn't matter)	Permutations, $_nP_r$ (order matters)

For $n = 5$, $r = 2$ of $\{A, B, C, D, E\}$:

$_5C_2 = 10$	$_5P_2 = 20$
AB	AB BA CA DA EA
AC BC	AC BC CB DB EB
AD BD CD	AD BD CD DC EC
AE BE CE DE	AE BE CE DE ED

$$_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{5!}{2!3!} = 10$$

$$_nP_r = \frac{n!}{(n-r)!} = \frac{5!}{3!} = 20.$$

$\binom{n}{0}$ — There is only 1 way to choose nothing

$\binom{n}{n}$ — There is only 1 way to choose everything

$\binom{n}{1}$ — There is only n ways to choose 1 thing

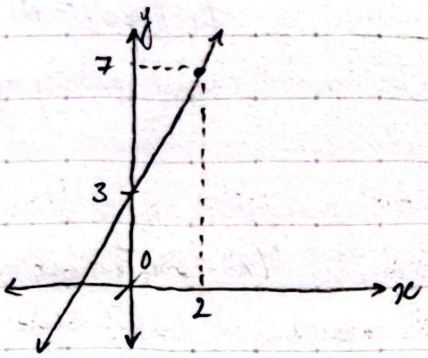
$\binom{n}{n-1}$ — Converse of $\binom{n}{1}$ → binomial theorem

Topic 5: Calculus

5.1 The Limit, $\lim f(x)$ and the derivative, $f'(x)$.

→ The Limit: $\lim f(x)$

Consider the function: $f(x) = 2x+3$



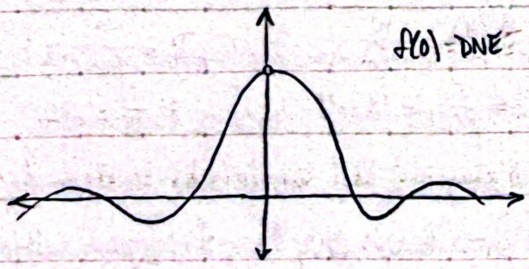
Let us investigate how the function behaves at at $x=2$. Clearly $f(2)=7$. But what happens when x is very close to 2.

x approaches 2^-	x approaches 2^+
(from values less than 2)	(from values more than 2)
if $x \to 2^-$ then $f(x) \to 7$	if $x \to 2^+$ then $f(x) \to 7$
$\lim_{x\to 2^-} f(x) = 7$	$\lim_{x \to 2^+} f(x) = 7$

$$\lim_{x\to 2} f(x) = 7$$

The situation $\lim_{x\to a} f(x) = f(a)$ occurs very often, but in cases where you can't substitute: e.g.

→ $f(x) = \frac{\sin x}{x}$ is not defined at $x=0$:

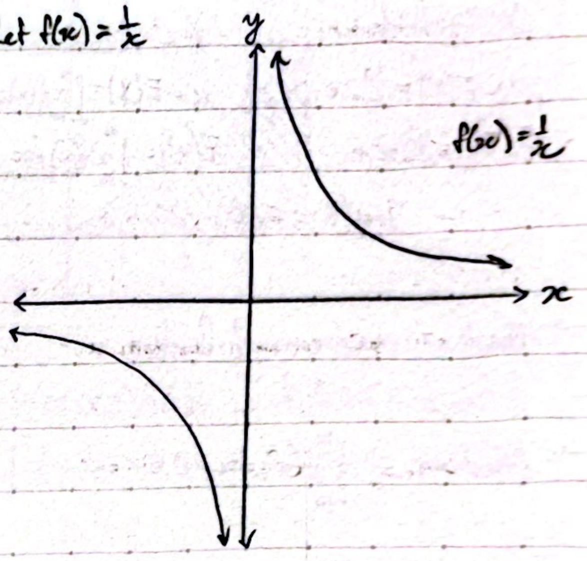


However as the limit as $x \to 0$ of $f(x) = 1$.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

The limit can be ±∞

Let $f(x) = \frac{1}{x}$



At $x=0$ $\lim_{x\to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x\to 0^+} \frac{1}{x} = +\infty$

$\lim_{x\to\infty} \frac{1}{x} = 0$ and $\lim_{x\to -\infty} \frac{1}{x} = 0$

In general if $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$ (or both)

$x=a$ is a vertical asymptote.

In general if $\lim_{x\to\pm\infty} f(x)$ and it converges to a $y=a$ is a horizontal asymptote.

- Exponential functions have a horizontal asymptote,
- logarithmic functions have a vertical asymptote.

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e = 2.7182818\ldots$$

→ Rate of Change (or Gradient) in a Straight Line

$$\frac{\Delta y}{\Delta x} = m = \cancel{df}\, f'(x) = \frac{y_2-y_1}{x_2-x_1}$$

↳ however this definition requires two points.

To get the instantaneous slope use the derivati

→ Derivative: $f(x)$, $\frac{dy}{dx}$

$f'(x)$ = Derivative = Rate of Change = Gradient at x

5.2 Derivative of Known Functions – Rules.

The derivative of a function, $f(x)$ is denoted by $f'(x)$

It indicates the rate of change of y in respects to x (otherwise known as the gradient). In general:

$$f(x) = x^n \quad f'(x) = nx^{n-1}$$

e.g.	x^{10}	$10x^9$
	x^4	$4x^3$
	x^3	$3x^2$
	x^2	$2x$
	x	$1x^0 = 1$
	100	0
	1	0

Some other common derivatives:

$\sin x$	$\cos x$
$\cos x$	$-\sin x$
e^x	e^x
$\ln x$	$\frac{1}{x}$
$\sqrt{x} = x^{\frac{1}{2}}$	$\frac{1}{2\sqrt{x}} = \frac{1}{x}x^{-\frac{1}{2}}$
c (constant)	0

Derivatives also follow for all rational (+, −) numbers

$x^{6.4}$	$6.4x^{5.4}$
$x^{\frac{3}{2}}$	$\frac{3}{2}x^{\frac{1}{2}}$
$x^{\frac{1}{2}}$	$\frac{1}{2}x^{-\frac{1}{2}}$

This also applies for non rational powers:

x^e	ex^{e-1}
$x^{2\pi}$	$2\pi x^{2\pi-1}$

→ Notation:

y'. $f'(x)$, $\frac{dy}{dx}$, $\frac{d}{dx}f(x)$ = derivative

or $f^{(n)}(x)$

The derivative at a specific x value:

$f'(2)$ or $\left.\frac{dy}{dx}\right|_{x=2}$

→ Rules of Differentiation (Algebraic)

- $(f+g)' = f' + g'$ $\quad (x^3 + 2x^2)' = 3x^2 + 4x$
- $(f-g)' = f' - g'$ $\quad (x^3 - 2x^2) = 3x^2 - 4x$
- $(af)' = af'$ a = const. $\quad 5x^2 = 5(2x) = 10x$
- $(af + bg)' = af' + bg'$ $\quad$ (combination of ↑)

→ Product Rule

$(f \cdot g)' = f' g + g' f$ e.g: $f(x) = x^5$ $g(x) = \sin x$

$(f' \cdot g)' = (x^5)' \sin x + x^5 (\sin x)' = 5x^4 \sin x + x^5 \cos x$

→ Quotient Rule

$\left(\frac{f}{g}\right)' = \frac{f' g - f \cdot g'}{g^2}$ * Order of f and g matter here more than in product rule.

e.g $f(x) = x^3$ $g(x) = \sin x$

$$\left(\frac{f}{g}\right)' = \frac{(x^3)' \cdot \sin x - x^3 (\sin x)'}{\sin^2 x} = \frac{3x^2 \sin x - x^3 \cos x}{\sin^2 x}$$

→ Higher Derivatives:

f(x) can be differentiated multiple times (to the n^{th} degree) to find the rate of change of the rate of change ...:

$f(x) = x^5$ $f'(x) = 5x^4$ $f''(x) = 20x^3$ ...

Polynomial functions always converge as $n \to \infty$ (or simply the deg of the function).

However, not all functions converge: If degree of function is <0 then the power of x approaches $-\infty$. If sin or cos, the function loops. If exponential, the differentiation is multiplied by the same factor:

$f(x) = \frac{1}{x} = x^{-1}$, $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, $f'''(x) = -6x^{-4}$...

$g(x) = \sin x$, $g'(x) = \cos x$, $g''(x) = -\sin x$, $g^{(3)}(x) = -\cos x$...

$[\ h(x) = e^{x}, h'(x) = e^{x}, h''(x) = e^{x}\ ]$ ↖

Special case: $u(x) = x$ for CR

→ Alternative Notation For Higher Derivation:

- $f''(x)$ can be written as $f^{(2)}(x)$, $\frac{d^2y}{dx^2}$, $\frac{d^2}{dx^2}f(x)$
- $f'''(x)$ can be written as $f^{(3)}(x)$, $\frac{d^3y}{dx^3}$, $\frac{d^2}{dx^3}f(x)$

5.3 The Chain Rule:

The chain rule states that if x is replaced with some other function, $u(x)$ then the derivative is of whatever manipulation is made to x but also multiplied by $u'(x)$:

$\sin u$	$(\cos u)(u')$
$\cos u'$	$(-\sin u)(u')$
e^{u}	$(e^{u})(u')$
$\ln u$	$(\frac{1}{u})(u')$

etc.

*This is how nested functions are differentiated.

*The chain rule can be applied multiple times:

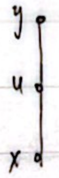
Let $f(x) = \ln(\sin(3x+1))$

$f'(x) = \frac{1}{\sin(3x+1)}[\sin(3x+1)]$ $\quad u = \sin(3x+1)$

$= \frac{1}{\sin(3x+1)}[3\cos+1]$ $\quad v = 3x+1$

$= 3\cot(3x+1)$

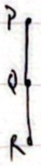
→ An Alternative Way to View The Chain Rule:



The chain rule says: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

→ HL Example:

Let $P = Q^3$ and $Q = \ln R$. Find $\frac{dP}{dR}$ in terms of R:



$$\frac{dP}{dR} = \frac{dP}{dQ} \cdot \frac{dQ}{dR} = 3Q^2 \cdot \frac{1}{R} = 3(\ln R)^2 \cdot \frac{1}{R}$$

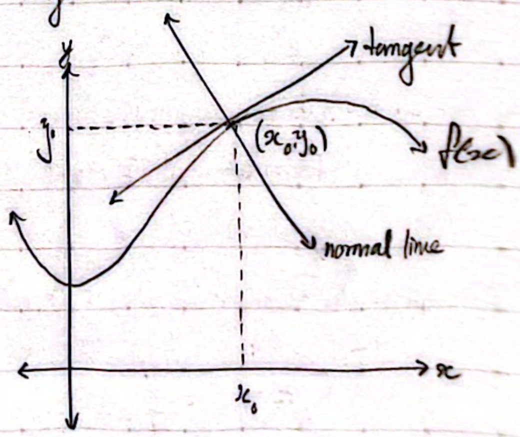
5.4 Tangent Line - Normal Line at Some Point x_0

Recall that straight lines that have a gradient m, pass through the point (x_0, y_0) has the equation

$y - y_0 = m(x - x_0)$ or

$y = mx + b$ where b accounts for (x_0, y_0)

→ Tangent and Normal Lines:



(x_0, y_0) = point of contact.

Tangent line at x_0: the line with gradient m_T that passes through (x_0, y_0). Normal line is perpendicular to $f(x)$, $m_N = \frac{-1}{m_T}$.

great job!

→ Methodology

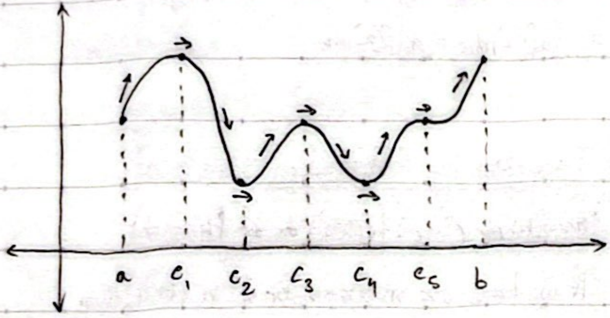
Given $y = f(x)$ and some point $x = x_0$

We find the point of contact (x_0, y_0), since $y_0 = f(x_0)$,

$f'(x) \rightarrow m_\perp f'(x_0)$ and so on $m_N = -\frac{1}{m_T}$

5.5 Monotony : Max, Min

→ Increasing / Decreasing Functions (Monotony)



The domain of the graph is the interval $[a, b]$ where $x = a$ and $x = b$ are the endpoints.

Local maxes : $x = b, c_1, c_3$

Local mins : $x = a, c_2, c_4$

To determine whether $f'(x) = 0$ is a min, max or point of criticality.) make a sign chart for the derivative: e.g:

$f(x) = \frac{1}{3}x^3 - 2x^2 + 3x + 5$ $f' = x^2 - 4x + 3$

solve for $0 = x^2 - 4x + 3 \rightarrow x = 1, 3$

x		1		3	
$f'(x) = x^2 - 4x + 3$	+	|	−	|	+
Nature of $f(x)$	↗	max	↘	min	↗

If $f'(x) > 0$ it is increasing if $f'(x) < 0$ its decreasing

However, if $f'(c) = 0$, it is a point of criticality

If $f''(c) > 0$ then c is a min, if $f''(c) < 0$, then c is a max)

↳ however this is known as concavity.

→ Concavity

$f''(x) > 0$ then f_1 is concave up (∪)

$f''(x) < 0$ then f is concave down (∩)

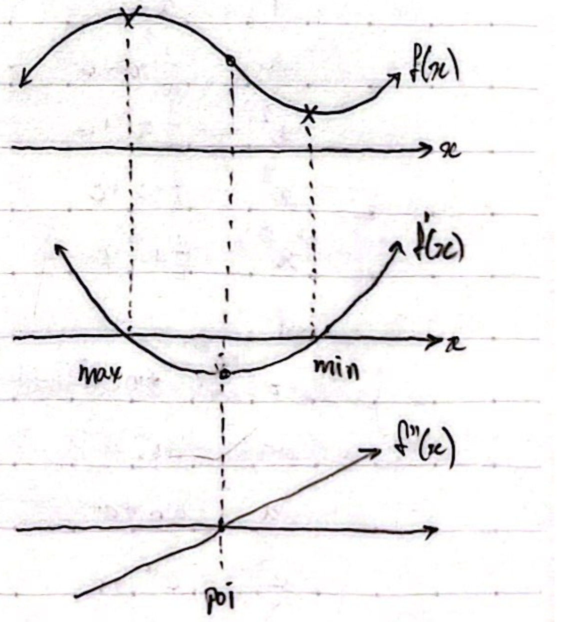
Points of inflection: $f''(d) = 0$

x		a		b		c	
$f''(x)$	+		−		+		+
	∪		∩		∪		∪
Conclusion for f		poi		poi		nothing	

5.7 The Graph of $f'(x)$

If we know the graph of $f(x)$ we can roughly sketch the graph of $f'(x)$. What we need is:

- Spot the stationary points; they become roots
- Increasing sections have a positive $f'(x)$ (above x axi
- Decreasing sections have a negative $f'(x)$ (below x ax



5.8 Optimization:

An alternative way to find the vertex of a function (quadratic) is to derived $f(x)$ and find $f'(x)$'s roots.

In problems of optimization, we have to construct a function in terms of some variable x. Then derive that constructed function.

5.9 The Indefinite Integral: $\int f(x)dx$

The integral or antiderivative finds is the inverse opperation for a derivative:

$$\int f'(x)dx = f(x).$$

In general:

$$F'(x)=f(x) \text{ then } \int f(x)dx = F(x)+c$$

The reasen some constant is added is because $c' = 0$ and $0 + F'(x) = f(x)$.

→ Common Integration Values:

$f(x)$	$\int f(x)dx$
1	$x+c$
x	$\frac{1}{2}x^2+c$
x^2	$\frac{1}{3}x^3+c$
x^3	$\frac{1}{4}x^4+c$
x^{10}	$\frac{1}{11}x^{11}+c$

∴ in general:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \quad \text{if } n \neq 1$$

→ Some other integrations:

a	$ax+c$
e^x	e^x+c
$\sin x$	$-\cos x+c$
$\cos x$	$\sin x+c$
$\frac{1}{x}=x^{-1}$	$\ln x+c$ $\hookrightarrow \ln\lvert x\rvert+c$

Must exist in $\ln x$'s domain.

→ Integrals for negative integers and rational numbe

$$\frac{1}{x^2}=x^{-2} \quad \int x^{-2}dx = -x^{-1}+c$$

$$\frac{1}{x^4}=x^{-4} \quad \int x^{-4}dx = -\frac{1}{3}x^{-2}+c$$

$$\sqrt[3]{x^2}=x^{\frac{2}{3}} \quad \int x^{\frac{2}{3}}dx = \frac{3}{5}x^{\frac{5}{3}}+c$$

$$\frac{1}{\sqrt[5]{x^3}}=x^{-\frac{3}{5}} \quad \int x^{-\frac{3}{5}}dx = \frac{5}{2}x^{\frac{2}{5}}+c$$

→ Two Basic Rules for Integration

- $\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$
- $\int af(x)dx = a\int f(x)dx$

5.10 Integration by Substitution

→ The Linear Case: $\int f(ax)\,dx$ or $\int f(ax+b)$.

If we have ax or $ax+b$ for x in $f(x)$, then the $f'(x)$ must be divided by a so that when differentiated, a is accounted for e.g.

$$\int \cos(3x+5)dx = \sin u\, du = \frac{\sin(3x+5)}{3}.$$

$$u = 3x+5 \quad du = 3dx$$
$$dx = \frac{1}{3}du$$

→ Supplimentary Information on dx: Δx = small infinitesimally small change in x.

→ Method of substitution:

- Let $u=g(x)$,
- Then $\frac{du}{dx} = g'(x) \Rightarrow dx = \frac{du}{g'(x)}$
- Express the initial integral in terms of u and du
- Calculate the new integral.
- Replace $u=g(x)$ back into the result
- (Solve if a definite integral)

e.g. $I = \int 3x^2(x^3+5)^7 dx$ let $u = x^3+5$

$= \int u^7 du$ $\quad du = 3x^2 dx$

$= \frac{u^8}{8} + c$

$= \frac{(x^3+5)^8}{8} + c$

→ Calculation by Inspection

If confident with substitution; calculation by inspection can be used to quicken calculations:

Integral	Result	
$\int u' e^u dx$	$e^u + c$	where u
$\int u' \sin u x\, dx$	$-\cos u + c$	is a function
$\int u' \cos u x\, dx$	$\sin u + c$	of x.
$\int \frac{u'}{u} dx$	$\ln u + c$	
$\int u' \cdot u^n dx$	$\frac{u^{n+1}}{n+1} + c$	

Integration by substitution is also known as the anti-chain rule for this reason.

5.11 The Definite Integral - Area between curves.

The definite integral is defined by: $\int_a^b f(x)dx$ where a, b are $\in \mathbb{R}$, within the domain of $f(x)$. To calculate this:

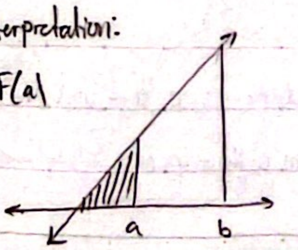
If $\int f(x)dx = F(x) + c$,

Then $\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$

→ Geometric Interpretation:

$F\Big|_a^b = F(b) - F(a)$

$=$



→ Properties of the Definite Integral.

If $\int_a^b [f(x)+g(x)] = \int_a^b f(x)dx + \int_a^b g(x)dx$

and $\int_a^b [c f(x)] dx = c\int_a^b f(x)dx$

then...

$\int_a^b f(x)dx + \int_b^c f(x)\overset{dx}{'} = \int_a^c f(x)dx$ (limits of a, b, c are consecutive)

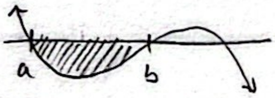
$\int_a^b f(x)dx = -\int_a^b f(x)dx$

$\int_a^b f'(x)dx = f(x)\Big|_a^b$

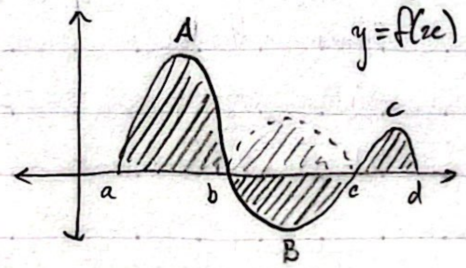
$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(y)dy \ldots$ variable x is irrelevant

→ Area Between Curve and x-Axis.

$\int_a^b f(x)$ is not always positive:



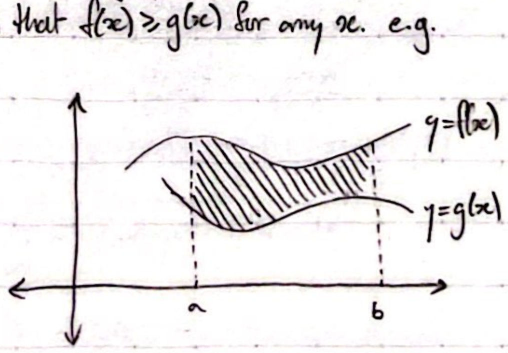
Thus to account for a function that has negative regions: the absolute of $f(x)$ must be integrated: $\int |f(x)| dx$.



Area of $\int_a^d f(x)dx = A - B + C = \int_a^d |f(x)| dx$

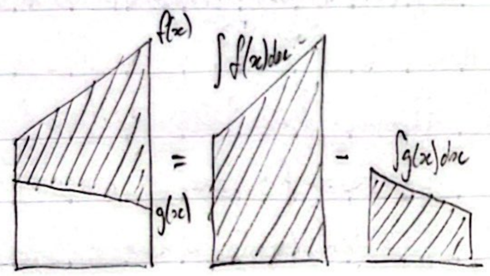
→ Area Between Two Curves:

Consider the graphs of two functions $y = f(x)$ and $y = g(x)$ such that $f(x) \geq g(x)$ for any x. e.g.



The area between f(x) and g(x) is given by:

$\int_a^b [f(x)-g(x)]dx$ (geometric explanation follows:)

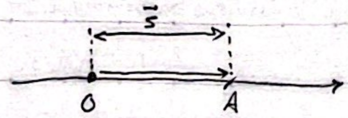


However if f(x) is not always $\geqslant g(x)$, then the area between two curves is defined by:

$$\int_a^b |f(x)-g(x)|dx$$

5.12 Kinematics: Displacement, Velocity, Acceleration

Consider the straight line and a fixed point O. A body is moving on the line (forwards or backwards). The displacement, $\vec{s}$ is the ± distance |OA| from the fixed point O.



The displacement as a function of time is given in meters (m). time (s).

The velocity = $\frac{d\vec{s}}{dt} = \vec{v}$ (m/s).

The acceleration = $\frac{d\vec{v}}{dt} = \frac{d^2\vec{s}}{dt^2}$ (m/s²)

$$\begin{tikzcd} \text{Displacement} \arrow[r, bend left, "\text{derive}"] & \text{Velocity} \arrow[r, bend left, "\text{derive}"] & \text{Acceleration} \\ \vec{s} & \vec{v} \arrow[l, bend left, "\text{integrate}"] & \vec{a} \arrow[l, bend left, "\text{integrate}"] \end{tikzcd}$$

The body is stationary at $\vec{v}=0$

The body is moving at constant velocity at $\vec{a}=0$

→ Displacement vs. Distance Travelled

Suppose that the velocity, $\vec{v}$ is given in terms of t

Displacement from O: $s=\int v\,dt$

Displacement from t_1 to t_2 $S=\int_{t_1}^{t_2} v\,dt$

Distance travelled from t_1 to t_2 $d=\int_{t_1}^{t_2} |v|\,dt$

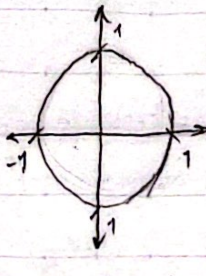
HL ONLY MATERIAL: TOPIC 5

5.13 More Derivatives (HL)

f(x)	f'(x)	
a^x	$a^x \ln a$	(make it into e^x)
$\tan x$	$\frac{1}{\cos^2 x} = \sec^2 x$	[quotient rule proofs for most.]
$\sec x$	$\sec x \tan x$	
$\csc x$	$-\csc x \cot x$	
$\cot x$	$-\csc x$	*All in data booklet
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$	
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$	Can be adapted for
$\tan^{-1} x$	$\frac{1}{1+x^2}$	← values other than 1

5.14 Implicit Differentiation - More Kinematics (HL)

In many real world applications, y cannot be solely expressed in terms of y: for one,



This relation has the relation:

$$x^2+y^2=1$$

To differentiate, this find the differential for each term with respects to the same variable:

e.g: $x^2+y^2=1$

$2x\cdot\frac{dx}{dx}+2y\frac{dy}{dx}=0$

$2y\frac{dy}{dx}=-2x$

$\frac{dy}{dx}=\frac{-2x}{2y}=-\frac{x}{y}$

↳ not defined at $x=0$

as it has an ∞ slope.

When given some point (x_0, y_0), the slope can be found for $x^2+y^2=1$.

→ The Derivative of an Inverse Function.

In general: $\frac{dx}{dy}=\left(\frac{dy}{dx}\right)^{-1}$

→ More on Kinematics

$a=\frac{dv}{dt}=\frac{dv}{dt}\cdot\frac{ds}{dt}=\frac{dv}{ds}\cdot v$ ← useful but derivable.

5.15 Related Rates (HL)

With rates that both depend on some variable t, if A is a function of t we know:

$\frac{dA}{dt}$ = rate of change for A

In general,

$$\frac{dy}{dt}=f'(A)\frac{dA}{dt}$$

We are given a relation between two quantities: A and B. Any change in A implies a change in B.

∴ There exists a relation between $\frac{dA}{dt}$ and $\frac{dB}{dt}$

Whenever we differentiate A we multiply by $\frac{dA}{dt}$, whenever we differentiate B we multiply by $\frac{dB}{dt}$

e.g: $A=2B^3 \rightarrow \frac{dA}{dt}=6B^2\frac{dB}{dt}$

$\sin A=\frac{3}{B} \rightarrow \frac{dA}{dt}\cos A=-\frac{3}{B^{\circ}}\frac{dB}{dt}$

→ Methodology

The problem usually refers to the rates of change for two quantities. One rate is given, one rate is required. (Usually at some constant)

1. Determine quantities A and B (say that $\frac{dA}{dt}$ is given and $\frac{dB}{dt}$ is required)
2. Find a general relation between A and B
3. Find the general relation between $\frac{dA}{dt}$ and $\frac{dB}{dt}$
4. If the question mentions a specific instant, for a specific value of B), Use step 2 to find A (if necessary) then substitute all known values into 3 to find $\frac{dB}{dt}$

→ Example:

Find the rate of change of h when $r=3$ and $h=6$ given two circumstances; and $A=\frac{1}{3}r^2h+2r^3$

a) when h is always double of r and $\frac{dA}{dt}=30$

since $h=2r$, the original relation becomes:

$A=\frac{2}{3}r^3+2r^3=\frac{8}{3}r^3$ Hence,

$\frac{dA}{dt}=8r^2\frac{dr}{dt}$ Therefore when $r=3$,

$30=72\frac{dr}{dt} \Leftrightarrow \frac{dr}{dt}=\frac{5}{12}$

b) when $\frac{dA}{dt}=30$ and $\frac{dh}{dt}=8$

By implicit differentiation, on the original relation,

$\frac{dA}{dt}=\frac{2}{3}rh\frac{dr}{dt}+\frac{1}{3}r^2\frac{dh}{dt}+6r^2\frac{dr}{dt}$ ∴ when $r=3$, $h=6$,

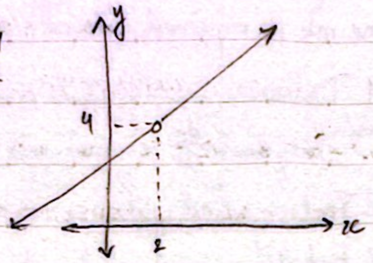
$30=12\frac{dr}{dt}+24+54\frac{dr}{dt}$

$6=66\frac{dr}{dt} \rightarrow \frac{dr}{dt}=\frac{1}{11}$

5.16 Continuity and Differentiability

Discontinuities are points or period for which the function is not defined. e.g:

$f(x) = \frac{x^2-4}{x-2}$



Not defined at $x=2$
+ not a VA

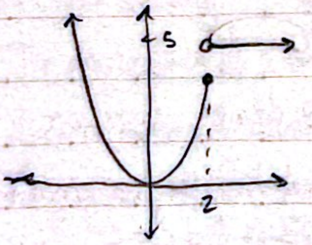
However as $x \to 2$, $y \to 4$. Thus $f(2)$ DNE, it's limit can be found instead.

$f(x) = \frac{x^2-4}{x-2} = \frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$ where $x \neq 2$

so $\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} x+2 = 4.$

We say that a function is continuous at $a=x$ if $f(a)$ exists, the $\lim_{x\to a} f(x)$ exists, and $\lim_{x\to a} = f(a)$

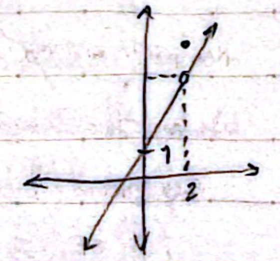
This is most often discussed with piecewise functions e.g.



$f(x) = \begin{cases} x^2 \text{ if } x \le 2 \\ 5 \text{ if } x > 2 \end{cases}$

$\lim_{x\to 2}$ does not exist as $\lim_{x\to 2^+} \neq \lim_{x\to 2^-}$

∴ not continuous.



$f(x) = \begin{cases} 2x+1 \text{ if } x \neq 2 \\ 7 \quad \text{ if } x = 2 \end{cases}$

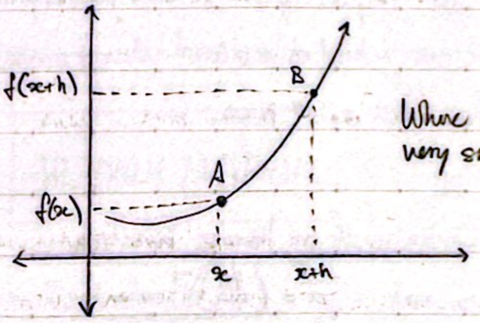
$\lim_{x\to 2}$ exists as the $\lim_{x\to 2^+} = \lim_{x\to 2^-}$

$(15=15)$ however $\lim_{x\to 2} \neq f(2)$ ∴

not continuous.

→ The Formal Definition of the Derivative:

Let $y=f(x)$ be a continuous and $A(x, f(x))$ be some point on $y=f(x)$.



Where h is very small.

As we move from point A to point B, the rate is:

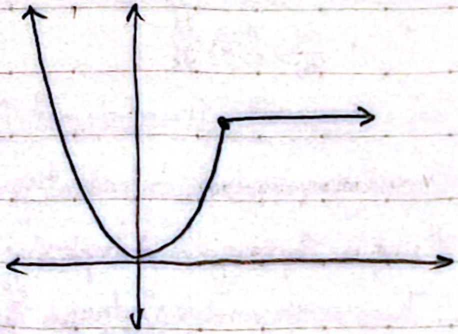
We select a neighboring point B with

x-coordinate = x+h

y-coordinate = f(x+h)

$\frac{\Delta y}{\Delta x} = \frac{f(x+h)-f(x)}{h}$ plug in any f(x)

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$



$f(x) = \begin{cases} x^2, \text{ if } x \le 2 \\ 4, \text{ if } x = 2 \end{cases}$ continuous at $x=2$,

not differentiable at $x=2$.

↳ In general if there is a cusp or sharp point ~~at~~ it is not differentiable.

5.17 L'Hôpital's Rule (HL)

→ Generalizations:

$\frac{0}{s} = 0$, $\frac{s}{0}$ DNE however if $0 \to x$, then

$\frac{x}{s} \to 0$, $\frac{s}{x} \to \pm\infty$ but what about $\frac{0}{0}$ and $\frac{\infty}{\infty}$?

Consider a function in the form: $\frac{f(x)}{g(x)}$

If $f(x) \to 0$, $g(x) \to 0$ the result can be anything: $0, \pm\infty$, or any $\in \mathbb{R}$. That is why we say $\frac{0}{0}$ and $\frac{\infty}{\infty}$ are indeterminant forms.

To find values at such points, L'Hôpital's Rule is used:

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

← and so on for n+1.

A commonly use limit/identity is $\frac{\sin x}{x}$ as

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \frac{1}{1} = 1.$$

→ Some Other Indeterminant Forms.

$$0 \cdot \infty, \infty - \infty, 1^{\infty}$$

5.18 More Integrals (HL)

$$\int a^x dx = \frac{a^x}{\ln a} + c$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \arctan \frac{x}{a} + c$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \arcsin \frac{x}{a} + c$$

$$\int \sec^2 x \, dx = \tan x + c$$

$$\int \operatorname{osc}^2 x \, dx = -\cot x + c$$

*The integrals of more trigonometric functions can be derived using integration by parts.

Integrals of the form

$$\int \frac{A}{ax^2+bx+c} dx \text{ or } \int \frac{A}{\sqrt{ax^2+bx+c}} dx$$

The methodology depends on the discriminant, Δ, but getting the denominator in the form $\sqrt{a^2-x^2}$ or a^2+x^2 will allow the function to be integrated using arcsin and arctan respectfully.

5.19 Further Integration by Substitution (HL)

In cases where the derivative is part of the integrand, e.g: $I = \int \frac{f'(x)}{f(x)} dx = \ln(f(x)) + c$.

$$\int e^{f(x)} \times f'(x) dx = e^{f(x)} + c$$

$$\int \sin(f(x)) \times f'(x) dx = -\cos f(x) + c$$

$$\int \cos(f(x)) \times f(x) dx = \sin f(x) + c$$

$$\int f(x)^n \times f'(x) dx = \frac{f(x)^{n+1}}{n+1} + c$$

* Do not forget this in questions involving $e^x \sin x$ or similar integrations.

5.20 Integration by Parts

This is what is used when functions are multiplied by each other (similar but distinct from the product rule; Integration by parts ≠ antiproduct rule!)

If the product rule is integrated, we get:

$$(u \cdot v)' = u' \cdot v + uv'$$

$$u \cdot v' = (u \cdot v)' - u' \cdot v$$

$$\int u \cdot v' dx = uv - \int u'v \, dx.$$

→ The Method.

Consider $I = \int x e^{x} dx$

Since $(e^{x})' = e^{x}$, the integral can be expressedas:

$$I = \int x(e^{x})' dx$$

Integration by parts gives

$$I = xe^{x} - \int x' e^{x} dx = xe^{x} - \int e^{x} dx = xe^{x} - x + c$$

However, there is an easier method:

derivative.

$$\int u \cdot v' dx = u \cdot v - \int u' \cdot v dx$$

integral

When integrating by parts, define u such that its derivative will cancel out or $\int u'v = I$, and so the integral of v isn't an issue:

Priority list for v' (i.e. for integration).

e^{x}, $\sin x$, $\cos x$

x^{n} (or polynomials)

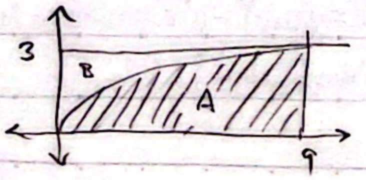
$\ln x$, $\arctan x$, $\arcsin x$, $\arccos x$.

→ Integration by Parts for Definite Integrals.

In the case of substitution, it would be safe to find the indefinite integral first and then the definite integral.

5.21 Further Areas Between Curves:

Consider the curve $y = \sqrt{x}$



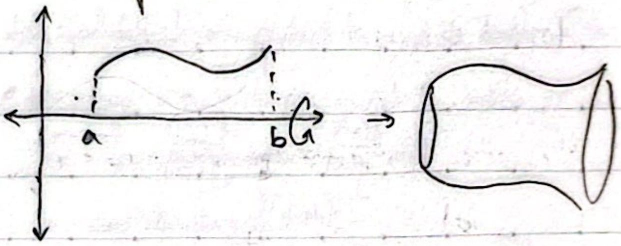
$A = \int_0^9 \sqrt{x} dx = 18$ $B = \int_0^9 (3 - \sqrt{x}) dx = 9$.

However if the area is defined in terms of y, then integrating will give the area about the y-axis. (By default it is about the x-axis)

In general, Area $= \int_a^b x dy = \int_a^b f^{-1}(y) dy$.

→ Volumes of Revolution:

If f(x) were rotated about the x-axis by 360°, 3D shapes can be modeled:



$$V = \pi \int_a^b (y^2) dx \rightarrow \pi \int_a^b (y_1^2 - y_2^2) dy = \pi \int_a^b x^2 dy.$$

5.22 Differential Equations:

$y = 2x^4 + 3$	particular solution
$y = 2x^4 + c$	general solution
$y(1) = 5$	boundary condition
$\frac{dy}{dx} = 8x^3$	differential equation

In general, a differential equation relates a function, $y = f(x)$ with its derivatives (x, y, y'', y''' etc.)

→ D.E. of Separable Variables.

A differential equation is of separable values if xs and ys can be separated in the form: $f(y)dy = g(x)dx$.

e.g. $\frac{dy}{dx} = 4xy^2$

$\frac{1}{y^2}dy = 4x\,dx$

$\int \frac{1}{y^2}dy = \int 4x\,dx$ collected c as a like term.

$-\frac{1}{y} = 2x^2 + c$

$y = -\frac{1}{2x^2+c}$

c can be solved using the boundary condition

→ Homogeneous D.E.

A differential equation is homogenous if it can take the form: $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ i.e. if RHS is a $f\left(\frac{y}{x}\right)$

e.g. $\frac{dy}{dx} = \frac{x^2-xy+y^2}{x^2} = 1 - \frac{y}{x} + \left(\frac{y}{x}\right)^2$

✱ Then the substitution $v = \frac{y}{x} \to y = xv$ is used such that

$$\frac{dy}{dx} = v + x\frac{dv}{dt} \quad \text{and}$$

$$v + x\frac{dv}{dt} = f(x)$$

→ 1st Order Linear D.E. (with Integrating Factor)

Differentials in the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where P(x) and Q(x) are functions of x only.

1. Find P(x) and Q(x)
2. Find the integrating factor: $I = e^{\int P(x)dx}$
3. It holds that $Iy = \int IQ\,dx$ (for y)
4. Calculate the intigral in the RHS and solve

e.g. $\frac{dy}{dx} + \frac{2}{x}y = 5x^2$ $P(x) = \frac{2}{x}$ $Q(x) = 5x^2$

The integrating factor: $I = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$

Then, $Iy = \int IQ\,dx \Rightarrow x^2y = \int 5x^4\,dx = x^5 + c$

$\therefore y = x^3 + \frac{c}{x^2}$

→ (Integration by Partial Fractions)

Sometimes f(x) in the form $\frac{px+q}{(x-a)(x-b)}$ (or something similar to Q)

To solve this use the following method:

S.No.		
1	$\frac{px+q}{(x-a)(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)}$
②	$\frac{px+q}{(x-a)^2}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2}$
3	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$
4	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$
5	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{(x-a)} + \frac{Bx+C}{(x^2+bx+c)}$

Where x^2+bx+c cannot be simplified further.

Depending on the equations equate the coefficients to their respective values:

e.g. $\frac{9x-7}{(3x+4)(x-5)} = \frac{A}{(3x+4)} + \frac{B}{(x-5)} = \frac{A(x-5)+B(3x+4)}{(3x+4)(x-5)}$

$\Rightarrow \frac{Ax+3Bx-5A+4B}{(3x+4)(x-5)}$

$A+3B = 9 \to A = 9-3B$

$-5A+4B = -7 \to -5(9-3B)+4B = -7$

$\to -45+15B+4B = -7 \to 19B = 38 \to B = 2$

$A+3(2) = 9 \therefore A = 3.$

When integrating something like $\frac{9x-7}{(3x+4)(x-5)}$ it can be broken into $\int \frac{3}{(3x+4)}dx + \int \frac{2}{(x-5)}dx$.

→ Euler's Method: Integral Approximation.

Consider the D.E.:

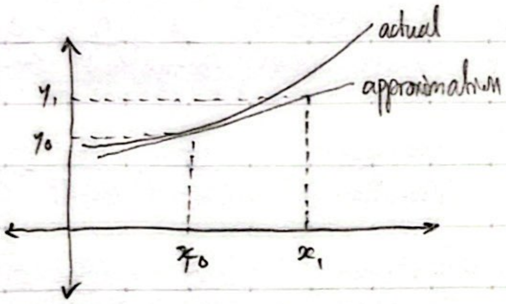
$$\frac{dy}{dx} = f(x,y) \quad y(x_0)=y_0$$

By using a step, h we find the consecutive points:

$$(x_0,y_0), (x_1,y_1), (x_2,y_2)\ldots$$

Where $x_{n+1} = x_n + h$

$$y_{n+1} = y_n + hf(x_n,y_n)$$



5.23 MacLaurin Series - Extention of Bino Thm (HL)

Consider the infinite geometric series:

$$1+x+x^2+x^3+\ldots$$

If $-1<x<1$ the sum converges to

$$S_\infty = \frac{1}{1-x}$$

MacLaurin Series seek to do the opposite:
Finding an infinite series defined by $f(x)=\frac{1}{1-x}$
= power series. in the form: $a_0+a_1x+a_2x^2\ldots$
(it resembles a polinomial with an infinite degree)

→ The MacLaurin Series.

Suppose that a function, f(x) has every order derivative near 0. Then f(x) is equal to:

$$f(x) = f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\ldots$$

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$ where $a=0$.

Changing a to be any number but 0. shifts the series by a, into a Taylor series.

→ Differential Equations and MacLaurin Series.

e.g $\frac{dy}{dx}=F(x,y)$ with a boundary condition $y(0)=y_0$
The analytical solution is not always possible so the MacLaurin Series can be used to approximate a solution for y=f(x).

→ Extension of the Binomial Theorem

$$(1+x)^n = \binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\ldots+\binom{n}{n}x^n$$

If the coefficients are expanded,

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots+x^n$$

This version allows for $n \in \mathbb{Z}$ and $\mathbb{Q}$

In the form $(a+b)^n=a^n\left(1+\frac{b}{a}\right)^n$, the binomial theorem becomes

$$(a+b)^n = a^n\left(1+n\left(\frac{b}{a}\right)+\frac{n(n-1)}{2!}\left(\frac{b}{a}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{b}{a}\right)^3+\ldots\right)$$

For this to converge $\left|\frac{b}{a}\right|<1$.

QED :)